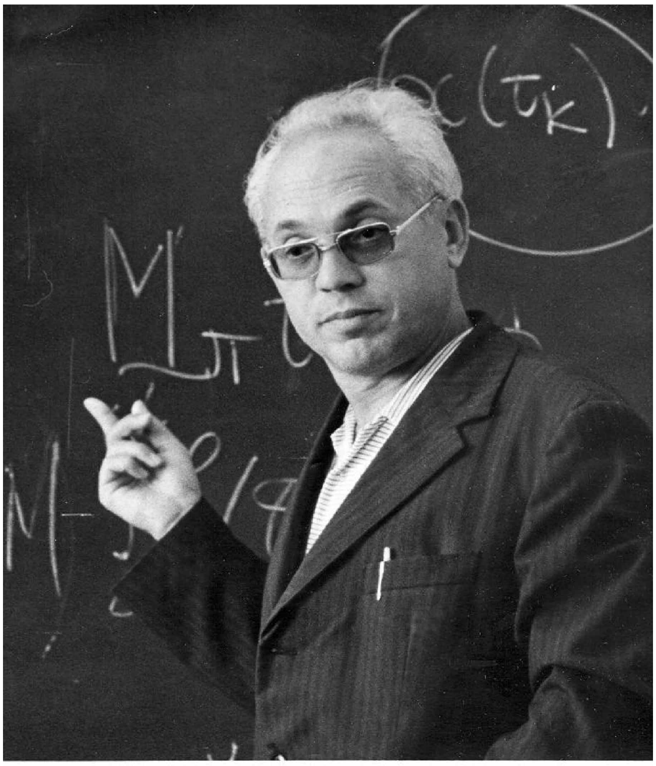
Anatolii Skorokhod (10.09.1930 – 3.01.2011)

Valerii Buldygin, Andrey Dorogovtsev, Mykola Portenko (Department of the Theory of Stochastic Processes, Institute of Mathematics National Academy of Sciences of Ukraine, Kyiv, Ukraine), Irina Kadyrova (Department of Mathematics, Michigan State University, East Lansing, USA)



This article is devoted to an outstanding mathematician and an excellent teacher A.V.Skorokhod, who has recently passed away. The introductory part was written by V.Buldygin (he died in 2012), A.Dorogovtsev, and M.Portenko. A version of Skorokhod's biography was presented by I.Kadyrova, who was his wife since 1975 up to his death. M.Portenko described his own impressions about the first book by A.V.Skorokhod, and A.Dorogovtsev presented his point of view on the evolution of the notion of the Skorokhod integral and related topics.

The name of A. V. Skorokhod belongs with those of the few outstanding mathematicians of the second half of previous century whose efforts have imparted modern features to mathematics. His extraordinary creative potential can be appraised by everyone who has studied the contemporary stochastic analysis and has revealed surprisingly the fact that the considerable part of its notions and methods has been introduced into mathematics by A. V. Skorokhod. It suffices to mention the notions of Skorokhod's topology, Skorokhod's space, Skorokhod's embedding problem, Skorokhod's reflecting problem, Skorokhod's integral and also the method of a single probability space, the notion of a strong and a weak linear random operator, the notion of a stochastic semigroup proposed as well by him and it becomes clear how extremely powerful was his creative capacity. Some of these notions (for example, the one of Skorokhod's integral) are now useful not only in mathematics but also in modern theoretical physics.

Being graduated from the University of Kyiv in 1953, A.V. Skorokhod carried on his post-graduate studies at the Moscow University within 1953-1956 and thereby he had the opportunity to acquire the achievements of the famous Moscow probabilistic school of those days with academician A. N. Kolmogorov at the head of it. Exactly there and then A.V. Skorokhod gained authority among the scientific world when he had managed to formulate and prove the general invariance principle. Some particular case of that principle was that time known as a result by M.Donsker established in1951. However, Skorokhod's result was not a simple generalization of that by M.Donsker. In order to formulate and prove it, A.V.Skorokhod introduced several new topologies into the space of functions without discontinuities of the second kind (one of those topologies is now well-known as Skorokhod's topology and it is useful in many branches of mathematics). Moreover, he proposed an original approach to the problem on the convergence of probability distributions (the method of a single probability space). Making use of those new tools, A.V.Skorokhod had formulated and proved the general invariance principle in an accomplished form. Those results are now included into any fundamental monograph on the theory of stochastic processes. The probabilists of those days were deeply impressed by new Skorokhod's ideas and the most authoritative probabilist, A. N. Kolmogorov, in 1956 published the paper "On Skorokhod's convergence" where he gave his own interpretation for the notions just introduced by A. V. Skorokhod. Professor of the University of California V. Varadarajan in one of his papers published in 2000 wrote: "I was a graduate student in Probability theory at the Indian Statistical Institute, Calcutta, India in 1956, and still remember vividly the surprise and excitement of myself and my fellow students when the first papers on the subject by Skorokhod himself and Kolmogorov appeared. It was clear from the beginning that the space D with its Skorokhod topology would play a fundamental role in all problems where limit theorems involving stochastic processes whose paths are not continuous (but are allowed to have only discontinuities of the first kind) were involved".

However, not only the famous Moscow school of Probability theory had an influence on the scientific work of A. V. Skorokhod. A significant part of his research was devoted to the theory of stochastic differential equations that had been originated by I.I.Gikhman, Kyiv in his works within 1940s-1950s (independently that theory arose in the works by K. Itô, Japan at about the same time). Not without an influence of I. I. Gikhman, A. V. Skorokhod was engaged into scientific investigations in that theory after his being back to Kyiv in 1957. The results of those investigations obtained by him within 1957–1961 formed the basis of his Doctoral dissertation and also his first book "Studies in the theory of random processes" published by the Kyiv University in 1961. The assertions expounded in that book as well as the methods used by A. V. Skorokhod for proving them were fundamentally different from those that were typical in the theory of stochastic differential equations by that time: the book was abundant in new ideas, new methods, and new results. In the beginning of 1960s A.V.Skorokhod published several articles devoted to the theory of stochastic differential equations that described diffusion processes in a region with a boundary. Those were the pioneer works and they stimulated a real stream of investigations on the topic at many probabilistic centers in the world. It should be said that the theory of stochastic differential equations has now become one of the most essential acquisitions of the whole mathematics of the second half of 20th century, and it is impossible to over-estimate the contribution of A.V.Skorokhod into it.

The full list of Skorokhod's publications consists of more than 300 articles published in various journals and 23 monographs, some of them written jointly with co-authors (the number of his monographs should be enlarged to 45 if their translations are taken into account). Under Skorokhod's supervising more than 50 graduate students defended their Candidate dissertations, 17 his disciples became the Doctors of mathematics. It should be added to this that A. V. Skorokhod paid considerable attention to popularizing mathematics among schoolchildren. He was a rector of the University of young mathematicians that during 10 years was in action at the Institute of Mathematics in Kyiv. Each academic year at that University started working with a lecture delivered by A. V. Skorokhod. He published 16 textbooks and popular-science books (some of them with co-authors).

A. V. Skorokhod was incessantly in quest of new mathematical truth. He was able to see the gist of a problem, to find out an original unexpected approach to it, and to create an adequate method for solving it. Besides, he had got into the habit of thorough thinking over every day. Owing to his intense work day after day, the creative spark given to him from God became now a bright shining star of the first magnitude on the mathematical frontier.

A brief biographical outline

Anatoli Vladimirovich Skorokhod was born September 10, 1930 in the town of Nikopol, Dnipropetrovsk region (in the past, Ekaterinoslavskaya province) in a family of teachers. Anatoli spent his childhood in southern Ukraine. His parents taught in rural schools around Nikopol. Anatoli's childhood took place during the very difficult 1930s: ruin after the Revolution and the Civil War of 1919–1922, collectivization of peasants, dispossession, exile, and hunger.

Anatoli's father, Vladimir Alexseevich, taught mathematics, physics and astronomy primarily in high school. Great teacher, erudite, he had a sharp analytical mind. From him Anatoli inherited an inquisitive, analytical mind, and a critical attitude towards everything. His father played a major role in the choice of his oldest son's profession. Anatoli's mother, Nadezhda Andreevna taught Russian and Ukrainian literature, history, music and singing, as well as mathematics. Nadezhda Andreevna had many different talents. She was a good musician, had a vivid dramatic talent. Nadezhda Andreevna also had good writing skills. Boasting an excellent style, she wrote scripts, stories, poems.

Anatoli entered elementary school at the age of seven. His studies were interrupted by World War II. The part of Ukraine, where the Skorokhods were living, was occupied in the beginning of the War.

The post-war years in the Southern Ukraine were years of poor harvest and in 1946, trying to escape from the hunger, the family moved to live in Kovel a town at Volyn, in the Western region of Ukraine. There the father was offered a position of a school principal. Studying in high school was easy for Anatoli, without any apparent effort. He was excellent in all subjects. Despite the fact that he was always interested in mathematics, during his school years Anatoli did not feel a predestination to become a mathematician.

After his graduation with a gold medal from high school in 1948, Anatoli following the advise of his father submitted his documents to the Kiev State University named after Taras Shevchenko, and was enrolled to the Faculty of Mechanics and Mathematics.

Skorokhod's scientific work began in his student years. Under the supervision of Boris Vladimirovich Gnedenko (at that time Chairman of the department of probability theory) and Iosif Illich Gikhman (then an associate professor of this department) Anatoli started his work in probability theory. At the end of his student years began Skorokhod's involvement in the research related to the famous Donsker invariance principle.

During 1953-1956 Anatoli was studying in graduate school of Moscow State University under the supervision of Eugene Borisovich Dynkin.

Years of study in the graduate school at the Moscow State University had been a remarkable period in Skorokhod's life in many ways. At that time (the 1950s) in the Faculty of Mechanics and Mathematics around the great teachers of the older generation gathered a broad audience of talented young people, who saw their future in the service of fundamental science. Among this group of young mathematicians. Anatoli Skorokhod was distinguished by his independence in the research work, courage and originality of approaches to problem solving. According to Anatoli, the main thing that he benefited from in Moscow graduate school was the seminar of his advisor E. B. Dynkin, called "Analysis, Algebra and Probability Theory".

The PhD thesis of Skorokhod (dissertation was defended in May 1957) contained descriptions of new topologies in the space of functions without discontinuities of the second kind and the application of them for proving limit theorems for stochastic processes.The Donsker invariance principle was generalized to the case when the limit process is a general process with independent increments. In the proofs of the theorems was used the original method invented by the author, known as the "method of a single probability space". The importance of the ideas of a very young mathematician was confirmed by the entire future development of the theory of stochastic processes. The terms "Skorokhod topology", "Skorokhod space" and "Skorokhod metric" are included in all basic books on the theory of stochastic processes.

In 1957-1964 Skorokhod was working as a faculty member in his "alma mater" – Kiev University. In 1961 in Kiev State University was published the first book by A.V.Skorokhod called "Studies in the Theory of Random Processes", which was the basis of his doctoral dissertation, defended in 1963. At the beginning of 1964 at the Institute of Mathematics of the Academy of Sciences of Ukraine was opened the Department of the Theory of Stochastic Processes and A.V.Skorokhod became a Head of this department. In the same year he was awarded the title of Professor. In 1967 A. V. Skorokhod was elected a corresponding member of the Ukrainian Academy of Sciences.

After Skorokhod's return from Moscow in 1957 began his friendship, scientific cooperation, and a further long-term and fruitful co-authorship with I.I.Gikhman. They wrote together many widely known books.

A.V.Skorohod played a prominent role in the development of Ukrainian, particularly Kiev school of probability theory. The scale and diversity of his research and teaching activities were striking. Generations of students grew up listening to his lectures, using his own written or co-authored textbooks and monographs.

Under A.V.Skorokhod leadership (since 1966) the national seminar on probability theory at Kiev State University has gained credibility and relevance not only in Kiev, but also far beyond. A.V.Skorokhod supervised graduate students at the University, as well as at the Institute of Mathematics. He was the advisor of 56 PhD students. Among his graduate students were not only Ukrainian students, but young scientists from India, China, Vietnam, East Germany, Hungary, Nicaragua and other countries. Under his consulting were also written 17 doctoral theses.

A. V. Skorokhod made great impact on raising the level of teaching elementary mathematics in Ukraine and popularization of mathematics.

In 1993–2011 Skorokhod worked at the Department of Statistics and Probability of Michigan State University, USA. His research areas were investigation of the behavior of dynamic systems under random perturbations, some problems of financial mathematics, and martingale theory. In 2000 Skorokhod was elected a member of American Academy of Arts and Sciences.

In an interview Skorokhod was asked how he felt about social activities of the scientists. To that Anatoli replied: "Negatively. I believe that a scientist should be a professional." However, under the circumstances that required demonstrating personal courage and providing support by his authority the actions in defense of the civil rights and freedoms of citizens, he joined the protesters. It happened in April of 1968 when a group of 139 scientists, writers and artists, workers and students wrote a letter to the leaders of the former USSR expressing their concern regarding the renewed practice of closed political trials of the young people from the midst of the artistic and scientific intelligentsia. The participation in this event was natural for A.V.Skorohod – a man with a sense of dignity, courageous, independent, who was never afraid of any authority and could not be indifferent to the flagrant flouting of civil rights in the country.

A V. Skorokhod was a true patriot of Ukraine. He hated everything that was part of the concept of "imperial thinking" with regard to Ukraine: denial of identity of language and culture, and of the very existence of the distinctive Ukrainian nation, the rejection of the idea of an independent Ukrainian state. That love for Ukraine made him an active participant of national liberation movement "People's Movement of Ukraine" ("Rukh", the late 1980s). Anatoli took part in all activities carried out by the initiative group when the movement was still only emerging. At that time the active participation in the creation of "Rukh" was dangerous, but that did not stop Skorokhod. When independence of Ukraine was proclaimed, and the "Rukh" began to turn into a political bureaucratic organization, Skorokhod completely lost interest in it and participation in its activities.

In mathematics phenomenon "Skorokhod" manifested itself throughout wideness and significance because besides mathematical talent Anatoli possessed equivalent gift of personality. His mathematical talent, intuition, efficiency caused surprise and delight, his modesty, indifference for awards and titles, absence of vanity, the independence of his judgments, the inner freedom served as moral standards in his social circle. Skorokhod's heartwarming subtlety and depth attracted people to him. Erudition in various fields of knowledge, history, philosophy, love of literature and classical music, passion for poetry (Skorokhod knew by heart the whole volumes of poems of his favorite poets Ivan Bunin, Osip Mandelshtam, Anna Akhmatova, Boris Pasternak, Joseph Brodsky, and was able to recite them for hours) caused an inspiration to follow his example, involved his friends and disciples in the same area of spiritual and aesthetic interests. Anatoli was a caring, loving son, loyal, understanding father, good friend, always ready to support in difficult circumstances, to listen, and to help. In personal relations he was very sincere, very romantic person, able to love selflessly.

In one of his articles, Anatoli wrote: "Only a curious to oblivion person can be a good mathematician.... With the help of mathematics new surprising and unexpected facts are often discovered. In fine art the beautiful creation always contains something unexpected, though not all unexpected is beautiful. Whereas in mathematics unexpected is always beautiful... there is nothing more beautiful than a simple and clear proof of a non-trivial statement". The engagement in mathematics was for Skorokhod a way of existence as natural as breathing.

"I think about mathematics always" – Skorokhod wrote in one of his letters. The hum of problems he thought about was continuous and incessant in his mind. In his work on problems Skorokhod did not dig deeply in the literature in the search of a suitable tool which could be adapted or modified to suit his needs. He created his own original methods and constructions that determined new directions in the development of the theory of stochasic processes for decades. Until the very end of his creative life Skorokhod maintained inquisitive curiosity, searching for harmony and beauty of mathematics.

Anatoli Vladimirovich Skorokhod died in Lansing, Michigan, January 3, 2011. Relatives and friends made last farewell with Anatoli with words: "A bright star has returned to the Universe". The ashes of Anatoli Skorokhod were buried May 20, 2011 at Baikove cemetery in Kiev.

A few words about the first book by A. V. Skorokhod

I was a fourth-year student at the Kyiv University, when the first book by A. V. Skorokhod "Studies in the theory of random processes" was published (1961). By that time I had already taken the course on the theory of stochastic differential equations taught by him, and that fact facilitated my efforts in comprehending the book. Even so, to read it was an uphill work for me, but I didn't give up reading it. Moreover, I managed to read it, while serving in the Soviet Army (1964-1965).

More than 50 years have passed since then. Many other books on the topic have been published by various authors (including Anatolii Volodymyrovych himself). However, for many researchers of my generation, the first book by A. V. Skorokhod still remains being a spark that has stimulated their enthusiasm for the theory of stochastic processes. The power of Skorokhod's creative ability and the courage of his searching mind were displayed in that book as brilliantly as five years before, in his fundamental work "Limit theorems for stochastic processes" (1956). I will consider some aspects of the book in this article.

With more than 50 years lasting experience in reading mathematical works by A. V. Skorokhod, I should say that besides usual difficulties felt by everyone when trying to comprehend anything new, some troubles connected with Skorokhod's manner to expound the material arose. In my opinion, Anatolii Volodymyrovych was not always irreproachable in this respect.One can sometimes come across a sentence in his texts that can be treated in several different ways, and it is difficult (particularly, for young mathematicians) to perceive which one belongs to the author. I once pointed out this carelessness that occasionally existed within his style. He replied that he was not able to understand why a reader should put into a written sentence a sense other than the author did.But I was not going to give in and said:"It is author's responsibility to structure any phrase in such a way that makes it clear for any reader what the author has meant." He replied with the question: "Do you really believe it possible to read a mathematical text without thinking over it?" I then understood his position. He showed no concern for particularizing materials expounded in his works. To think over new problems was more important for him than to expound the thoughts and the ideas that have already been discovered by him or somebody else.

There is another source for difficulties in reading the texts by A.V. Skorokhod. According to his own confession, if he made a mistake when writing a sentence or a formula, he did not notice it when re-reading that sentence or that formula: instead of what was written he saw what should be written.

I conclude this introductory part of my article with the words: Skorokhod's texts are not straightforward, but it is worth it to read them.

* * *

A. V. Skorokhod started lecturing at the Kyiv University in 1957. Within previous three years he was a postgraduate student at the Moscow University. His studies there were completed with a dazzling success: he had formulated and proved the most general limit theorems for stochastic processes and, moreover, he had invented an original method for proving them. In spite of his young age, he had already succeeded in getting authority among experts in Probability theory. His PhD thesis (Candidate dissertation) had already been prepared for defense and rumours were afloat that the second scientific degree, i.e. Doctorate of mathematics, would be conferred to him for that thesis (in reality it turned out to be not so for some reason that will not be discussed here).

In such a situation it would be natural for everyone to take a pause in scientific research, but not for Anatolii Volodymyrovych. His list of publications shows that his searching mind was working incessantly. However, the main field of his scientific interests was changing: the theory of stochastic differential equations started attracting his attention. This theory had been just originated by the early fifties. It would unlikely be right to speak of its details here. I only say that the theory was independently created by K. Itô (Japan) and I. I. Gikhman (Kyiv) in their works published during forties and early fifties of the previous century.

K. Itô developed the theory of stochastic differential equations basing on the notion of a stochastic integral introduced by him. His notion was a generalization of Wiener's one in two directions: first, the integrand in his notion was a random function (Wiener constructed the integral of a non-random one), and second, he constructed integrals not only with respect to a Brownian motion, but also with respect to a (centered) Poisson measure.Given some local characteristics of a stochastic process to be constructed and, besides, such "simple" objects as a Brownian motion and a Poisson measure, he wrote down a stochastic integral equation (it could be written as a differential one) whose solution gave the trajectories of the process desired. Under some conditions on given coefficients (the local characteristics mentioned above), he managed to prove the existence and uniqueness of a solution and to establish its property of being a Markov process. The set of stochastic processes that were differentiable in Itô's sense was endowed by a calculus different from that of classical type (for example, the Itô formula created a new rule of differentiating a function of a stochastic process having Itô's stochastic differential). The approach to the theory of stochastic differential equations given by K. Itô turns out to be exceptionally proper: in the most of monographs on the topics, the basic notion is the notion of Itô's integral or some its generalizations.

I.I. Gikhman was not possessed of such a notion. Nevertheless, his notion of a stochastic differential equation was quite rigorous in mathematical sense. It was based on the notion of a random field that locally determined the increments of the process to be constructed as a solution to the corresponding stochastic differential equation. Under some conditions on a given random field, I. I. Gikhman proved the theorem on the existence and uniqueness of a solution with a given initial data. If the random field did not possess a property of aftereffect, then the solution was a Markov process. In the case of that field being given by a vector field of macroscopic velocities plus the increments of a Brownian motion transformed by a given operator field, the corresponding solution turned out to be a differentiable function with respect to initial data (under the assumption, of course, that the mentioned fields were given by smooth functions in spatial arguments). Having this result, I. I. Gikhman managed to prove the theorem on the existence of a solution to Kolmogorov's backward equation (i.e., a second order partial differential equation of parabolic type) without any assumptions on the non-degeneracy property of the matrix consisting of the coefficients of the second spatial derivatives (it is well-known how important is such a property in the analytical theory of those equations). That was a significant result showing that some theorems in the theory of partial differential equations can be proved with the use of purely probabilistic methods.

I have just described briefly the situation in the theory of stochastic differential equations formed by the beginning of 1957. I think that A. V. Skorokhod had the opportunity to acquaint himself with the Itô theory during his studies in Moscow. As far as I know, just after his being back to Kyiv, his regular discussions with I. I. Gikhman started taking place and he was able to comprehend Gikhman's approach to the theory. That branch of mathematics was then quite new and A. V. Skorokhod was entirely engaged into investigations on the topic. His works published within 1957–1961 (though not all of them) were related to the theory of stochastic differential equations. At the end of 1961 in the publishing house of the Kyiv University the first book by A. V. Skorokhod came out. Besides the title, there was a subtitle on its cover, namely "Stochastic differential equations and limit theorems for Markov processes".

A. V. Skorokhod started the book with setting the Itô theory of stochastic differential equations forth (more precisely, its multidimensional version) and then he presented several of his new results that essentially influenced the further evolution of the theory itself.

First, he proved the theorem on the existence of a solution to a stochastic differential equation under the assumption that its coefficients were only continuous (they also were assumed to satisfy the usual growth conditions at infinity, of course), i.e. they might not satisfy the Lipschitz condition in spatial arguments, as was in Itô's theory or Gikhman's one. That theorem was an analogue to the Peano theorem in the theory of ordinary differential equations. The uniqueness of a solution was not guaranteed. It should be also said that the solution constructed by A. V. Skorokhod in proving that theorem seemed to be of a somewhat different character from those constructed by K. Itô or I. I. Gikhman. The former and the latter constructed the solution on the probability space where an initial position, a Brownian motion, and a Poisson measure (or Gikhman's random field) were given and that solution turned out to be a functional of those objects. Solutions with this property came later to be called strong solutions. And A. V. Skorokhod made use of the method of a single probability space (invented by him before) for proving his theorem and it could not be guaranteed that his solution was a strong one. An absorbing problem then arose: what conditions on given coefficients of a stochastic differential equation one should impose in order to assert that the solution of that equation were strong. A detailed investigation of that problem can be found in the fundamental article by A. K. Zvonkin and N. V. Krylov "On strong solutions of stochastic differential equations" (1974).

Second, for stochastic differential equations describing the processes with jumps, he established the differentiability of a solution with respect to the initial data. This allowed him to derive an integro-differential equation for the corresponding mathematical expectation. That equation was an analogue to the Kolmogorov backward equation for diffusion processes obtained before by I. I. Gikhman (see above).

Third, he found out the conditions under which a pair of stochastic differential equations generated two measures on the space of all functions without discontinuities of the second kind such that one of those measures was absolutely continuous with respect to other one, and the formula for the corresponding density was written. Those formulae are very important in mathematical statistics when some unknown parameters in the coefficients are to be estimated or one of the two given competitive hypothesis about the coefficients is to be chosen.

Fourth, he formulated and proved a very interesting theorem on comparison of solutions to a given pair of stochastic differential equations on a real line whose diffusion coefficients coincided and whose drift coefficients were related with an inequality valid for all instants of time and at any point of the real line (a term containing a Poisson measure was absent in the equations under those considerations). Then it turned out that the solutions were related with the same inequality as ever, if only their initial positions did so. Making use of this theorem, A. V. Skorokhod established the uniqueness of a solution to a one-dimensional stochastic differential equation with continuous coefficients satisfying the usual growth conditions at infinity and such that the diffusion coefficient was given by a Hoelder continuous function (in spatial argument) with the exponent greater than 1/2. As a matter of fact, it was the first result on the existence of a strong solution to an equation with non-Lipschitzean coefficients.

Fifth, he found out the conditions on a given sequence of Markov chains such that the stochastic processes generated by those chains were weakly convergent (in the first of the topologies introduced by him into the space of all functions without discontinuities of the second kind) to the solution of a stochastic differential equation. Some results of the kind had been known by that time for the case of a limiting process being a diffusion one.

Sixth, the last Chapter of the book was devoted to the so-called "embedding problem "that could be formulated as follows: for a given one-dimensional Brownian motion, an integrable stopping time was to be constructed such that the value of the Brownian motion at that stopping time was distributed according to a beforehand given measure on a real line being centered and having finite second moment. A. V. Skorokhod solved this problem and gave its application to estimating the probability that a sequence of the normalized sums of independent random variables was located inside the region bounded by two given curves. The embedding problem formulated and solved by A.V. Skorokhod in 1960 has stimulated investigations in probability theory for over 50 years. On an International conference "Skorokhod space: 50 years on" that took place in Kyiv in 2007, one of its sections had the title "The Skorokhod Embedding Problem". The organizer of that section, Professor J. Obloj from UK, published an excellent brief review of given then talks on the topic: "The Skorokhod embedding problem: old and new challenges" (see Abstracts of that conference, part 1, pp. 93–97).

The points from first to fifth listed above were the first Skorokhod's steps in the theory of stochastic differential equations. It should be mentioned that there was one more his step in that theory made between the years 1957–1961, namely, his pioneer results related to the theory of stochastic differential equations describing diffusion processes in a region with boundary points. Those results were not included into the book. They originated the branch of the theory of stochastic processes called now "The Skorokhod Reflection Problem". A section with this title organized by Professors P. Dupuis and K. Ramanan from the USA was also working at the conference mentioned above.

Skorokhod's achievements described above show how extremely long his first steps were into the theory of stochastic differential equations, how quickly he had covered the way from the position of a beginner to the one of a leader. That theory became his favorite field of mathematics and he had many opportunities after 1961 to think over some of its problems again and again.

Many people knew how great he was not only as a mathematician, but also as a human being. I would like to address anyone who was close to him with the following lines:

> Don't say wistfully: "He is no more..."
> But say thankfully: "He was!"

The evolution of the Skorokhod integral

One of the important notions of modern stochastic calculus is the notion of the extended (also called Skorokhod) stochastic integral. It arose at the beginning of 70s of XX century.

In the original Skorokhod paper [16], the extended integral was built in as an operator on Hilbert-valued Gaussian functionals. Suppose that H is a real separable Hilbert space and ξ is a generalized Gaussian random element in H with zero mean and identity covariation. If σ-field $\mathcal{F}$ of random events is generated by ξ, then every square-integrable random variable α can be expanded into the Itô-Wiener series

$$\alpha = \sum_{k=0}^{\infty} A_k(\xi, \ldots, \xi). \tag{1}$$

Here $\{A_k(\xi, \ldots, \xi)\}$ are the infinite-dimensional Hermite polynomials associated with the symmetric Hilbert–Shmidt forms on H.

$$E\alpha^2 = \sum_{k=0}^{\infty} k! \|A_k\|_k^2, \tag{2}$$

where $\|A_k\|_k$ is the Hilbert–Shmidt norm on $H^{\otimes k}$. Define the stochastic derivative $D\alpha$ of α as a square-integrable random element in H which satisfies the relation

$$(D\alpha, \varphi) = \sum_{k=0}^{\infty} kA_k(\varphi, \xi, \ldots, \xi),\ \varphi \in H. \tag{3}$$

Finally, the Skorokhod integral can be defined as the adjoint operator $I = D^*$ acting from the space of the square-integrable H-valued random elements to the space of square-integrable random variables. In the terms of Itô–Wiener expansion, the Skorokhod integral can be described as follows. Let the square-integrable random element x in H be represented by the series $(x, \varphi) = \sum_{k=0}^{\infty} A_k(\varphi, \xi, \ldots, \xi)$, $\varphi \in H$. It can be checked that $A_k \in H^{\otimes k+1}$ for every $k \geq 0$. Denote by ΛA_k its symmetrization. Then $I(x) = \sum_{k=0}^{\infty} \Lambda A_k(\xi, \ldots, \xi)$.

This algebraic approach to the definition of I and D gives a possibility to derive many useful properties of these operators and disclose deep relationships with quantum physics and leads to such results as hypercontractivity of Ornstein–Uhlenbeck semigroup, logarithmic Sobolev inequality etc (see [32, 35]).

The briefly described algebraic approach to the definition of the Skorokhod integral is not unique. Another one is closely related to the theory of Gaussian measures in infinite-dimensional spaces. Let the space H be imbedded into another Hilbert space H' by a Hilbert–Schmidt operator. Then ξ becomes a usual random element in H'. Let ν be the distribution of ξ in H'. Note, that H is the space of the admissible shifts for ν. Consider a function $f \in C^1(H')$ with bounded derivative. Then f has a bounded Fréchet derivative f'_H along H. It can be proved, that the stochastic derivative $Df(\xi)$ coincides with $f'_H(\xi)$. Taking the closure we can organize the Sobolev space $W_2^1(H', \nu)$ of the functions with the Sobolev derivative along H. In this construction, the extended stochastic integral $I(x)$ arises via the integration-by-parts formula for Gaussian measures and can be interpreted as a logarithmic derivative of the initial measure ν along the vector field (H-valued) x [38, 37]. Wiener measure the stochastic derivative and its adjoint operator were also introduced in [22]. The extended integral is used in measure theory. For example, the Radon–Nicodym density of the measure transformed by a flow of mappings can be described in terms of extended integral [15, 14, 36]. On the other hand, the relationship with measure theory gives a possibility to construct the Skorokhod integral in non-Gaussian situation using logarithmically smooth measures, by differentiation along non-linear transformations and even the formal rules of differentiation on the space of random variables [7 – 11, 33, 34].

The Skorokhod integral got its name "extended" due to the most known case of its application, when $H = L_2([0; 1])$ and ξ is generated by the Wiener process w via the following formula $\forall\ \varphi \in H:\ \ (\varphi, \xi) = \int_0^1 \varphi dw$. It occurs, that the multidimensional Hermite polynomials $A_k(\xi, \ldots, \xi)$ are the multiple Wiener integrals $\int_0^1 \overset{k}{\ldots} \int_0^1 a_k(\tau_1, \ldots, \tau_k) dw(\tau_1) \ldots dw(\tau_k)$ with the symmetric kernels a_k, satisfying the relation $\int_0^1 \overset{k}{\ldots} \int_0^1 a_k(\tau_1, \ldots, \tau_k)^2 d\tau_1 \ldots d\tau_k = \|A_k\|_k^2$. Really, let x be a random element in $L_2([0; 1])$ with the finite second moment. x can be viewed as a random function. Define the filtration $(\mathcal{F}_t)_{t\in[0;1]}$ associated with the Wiener process as usual. Then suppose that x is adapted to $\mathcal{F}_t$ (as a random element in $L_2([0; 1])$). It is well-known [25, 5] that under this condition x belongs to the domain of definition of the Itô stochastic integral with respect to Wiener process. But it is very interesting that x belongs to the domain of I and $I(x) = \int_0^1 x(s)dw(s)$. So, the Itô integral is a partial case of the Skorokhod integral. This fact was found out in 70-th of XX-th century by many authors. Since then, many articles have been written on the Skorokhod integral and stochastic integration. The main directions of investigation here were the following. In the Itô case and in more general construction of the stochastic integral with respect to semimartingale, the approximation by the integral sums can be used. In particular, the Skorokhod integral was obtained via approximation by the step functions, Ogawa symmetric integral was created, the relationships between these integrals and Stratonowich integral were studied (see for ex. [13 – 17]). In the case of the Skorokhod integral with an indefinite upper limit, the quadratic variation remains the same as in Itô case and some interesting relations with martingales can be obtained [18 – 20, 32]. One of the main properties of the Itô integral is its locality. Namely, if random functions x_1, x_2 on $[0; 1]$ are integrable with respect to the Wiener process w, then $\Big(\int_0^1 x_1 dw - \int_0^1 x_2 dw\Big)\mathbf{1}_{\{x_1 = x_2\}} = 0$. This property is very important. For example, it allows to use the stopping technique in the consideration of the stochastic differential equations. The corresponding property of the Skorokhod integral was established in two different situations. In initial Skorokhod article, the following statement was proved: if the random element x has a stochastic derivative, then x lies in the domain of I. For stochastically differentiable random functions the Skorokhod integral has a locality property. This was proved in [19].

The other approach was proposed in [5, 2]. Define the smooth open subset Δ of probability space as $\Delta = \{\omega : \alpha(\omega) > 0\}$, where α is a stochastically differentiable random variable. Then it can be proved, that for x_1, x_2 from the domain of I (not necessarily stochastically differentiable) the equality $(x_1 - x_2)\mathbf{1}_\Delta = 0$ implies

$$(I(x_1) - I(x_2))\mathbf{1}_\Delta = 0. \tag{4}$$

The above mentioned statements give us a possibility to define the Skorokhod integral for the random functions that do not have finite moments. The recent results on the description of subset of probability space, which have a locality property (4) can be found in [21]. Stochastic equations with the non-Itô integrals have been actively studied since 70-th of the XX-th century. As an example the boundary value problems or integral equations of the second kind can be considered. Such equations were treated initially in the article [17, 1] using the algebraic definition of the Skorokhod integral. One of the interesting cases where anticipation arises is the Cauchy problem for ordinary stochastic differential equation with the initial condition which depends on the future noise. The interesting result was obtained in [29]. A linear one-dimensional equation was considered

$$\begin{cases} dx(t) &= a(t)x(t)dt + b(t)x(t)dw(t) \\ x(0) &= \alpha. \end{cases} \tag{5}$$

Here α is a functional of w and the equation is treated in the sense of the Skorokhod integral. It was proved in [29], that the solution has a form

$$x(t) = T_t\alpha \cdot \mathcal{E}_0^t, \tag{6}$$

where $T_t\alpha$ is transformation of α corresponding to the change $w(\cdot) \to w(\cdot) + \int_0^{t\wedge\cdot} b(s)ds$, and $\mathcal{E}_0^t$ is a usual stochastic exponent (related to the case $x(0) = 1$). The reason of appearance of $T_t\alpha$ in the solution can be seen easily if we recall the definition of the Skorokhod integral as a logarithmic derivative. It can be verified, that I has a structure of the infinite-dimensional divergence operator. So (5) can be treated as a first order differential equation in the infinite-dimensional space. Then (6) becomes to be the solution obtained via the characteristic method. This approach was introduced in [7]. This point of view on the equation (6) explains the absence of the solution in general non-linear case and the absence of the good form of solution in the linear vector case. The last case with the commuting matrix-valued b was considered in [30]. Surprisingly, the algebraic definition of the Skorokhod integral turns to be very helpful in the consideration of (6). The following representation of the solution was obtained in [8] $x(t) = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_0^t \overset{k}{\cdots} \int_0^t D^k\alpha(s_1, \ldots, s_k) \cdot \cdot \mathcal{E}_{s_k}^t b(s_k) \mathcal{E}_{s_{k-1}}^{s_k} b(s_{k-1}) \ldots \mathcal{E}_0^{s_1} ds_1 \ldots ds_k$. In the 90-th of the XX-th century a new reason for studing the extended stochastic integral arose. This was the development of the models from financial mathematics. In these models the fractional Brownian motion plays the role of the noise process [31]. So, the integration with respect to it must be constructed. One of the possible approaches leads to the extended stochastic integral with respect to the Gaussian integrators [9]. Gaussian process η on $[0; 1]$ is called the integrator if there exists such $c > 0$ that for an arbitrary partition $0 = t_0 < t_1 < \ldots < t_n = 1$ and real numbers $a_1, \ldots, a_n$ we have

$$E\Big(\sum_{k=0}^{n-1} a_k(\eta(t_{k+1}) - \eta(t_k))\Big)^2 \le C \sum_{k=0}^{n-1} a_k^2 (t_{k+1} - t_k). \tag{7}$$

In general setup, introduced by A.V.Skorokhod, the noise generated by the integrator η can be related to the Gaussian generalized element in H of the form $A\xi$ with the certain continuous linear operator A. So, now the definition of the integral with respect to η can be $I(A^*x)$, where I is the original Skorokhod integral. This construction is a particular case of the action of the random map on the random elements proposed in [5]. The corresponding Itô formula for the Skorokhod integral with respect to the Gaussian integrator was proved in [9]. Note that integrator does not necessarily have the semimartingale property. Hence, the related stochastic calculus is purely non-Itô calculus. In varions mathematical models where the anticipation arises, the type of the construction of the stochastic integral is motivated by the external reason. For example, in the stochastic boundary value problem physicists prefer the symmetric stochastic integral [33]. But the same problem can be treated with the Skorokhod integral (with the different solution of course) [1]. In spite of this, there exist mathematical problems where the extended stochastic integral arises naturally with necessity. These are the problems of the filtration theory. The main property of the Skorokhod integral here is the following

$$\Gamma(A) \int_0^T x(t)dw(t) = \int_0^T \Gamma(A)x(t)d\gamma(t), \tag{8}$$

where $\Gamma(A)$ is the operator of the second quantization in the space of Wiener functionals and the differential $d\gamma$ is the Skorokhod differential with respect to the integrator $\gamma = \Gamma(A)w$. As it is well-known, $\Gamma(A)$ can be an operator of conditional expectation in the case when A is a projector [32]. So the relation (8) can be used for deriving the anticipating equation for optimal filter in nonsemimartingale case (see [8, 10]). Last equation is a partial stochastic differential equation with anticipation. The properties of the solutions (existence, smoothness, large deviations) are studied in [13].

The above mentioned ideas and facts about the Skorokhod integral reflect the main steps in the development of this notion. More detailes can be found in the references. Certainly the list of reference is far from completeness. But I suppose that the main ideas were mentioned and the cited references can be used for the first studing and working with such a rich and beautiful object as the Skorokhod integral.

References

[1] A.A.Dorogovtsev. Boundary problem for the equations with stochastic differential operators. *Theory Probab. Math. Statist.*, 40(11):23–28, 1989.

[2] A.A.Dorogovtsev. Stochastic calculus with anticipating integrands. *Ukrainian Math. J.*, 41(11):1460–1466, 1989.

[3] A.A.Dorogovtsev. Stochastic integrals with respect to gaussian random measures. *Theory Probab. Math. Statist.*, 44:53–59, 1992.

[4] A.A.Dorogovtsev. One property of the trajectories of the extended stochastic integrals. *Sibirian Math. J.*, 34(5):38–42, 1993.

[5] A.A.Dorogovtsev. *Stochastic Analysis and Random Maps in Hilbert Space*. VSP, Utrecht, The Netherlands, Tokyo, Japan, 1994.

[6] A.A.Dorogovtsev. One approach to the non-caussian stochastic calculus. *J. Appl. Math, and Stoch. Anal.*, 8(4):361–370, 1995.

[7] A.A.Dorogovtsev. *Anticipating stochastic equations*. Proceedings of the Institute of Mathematics of the National Academy of Sciences of the Ukraine, 15. Institut Matematiki, Kiev, 1996.

[8] A.A.Dorogovtsev. Anticipating equations and filtration problem. *Theory Stoch. Proc.*, 3 (19)(1–2):154–163, 1997.

[9] A.A.Dorogovtsev. Stochastic integration and one class gaussian stochastic processes. *Ukrainian Math. J.*, 50(4):495–505, 1998.

[10] A.A.Dorogovtsev. Smoothing problem in anticipating scenario. *Ukrainian Math. J.*, 57(9):1218–1234, 2005.

[11] A.A.Dorogovtsev. Smoothing problem in anticipating scenario. *Ukrainian Math. J.*, 57(10):1327–1333, 2006.

[12] A.Benassi. Calcul stochastique anticipatif: Vartingales hierarchiques. *C.R. Acad. Sei., Ser.l, Paris*, 311(7):457–460, 1990.

[13] A.Millet, D. Nualart, and M.Sanz-Sole. Composition of large devistion principles and applications. *Ann. Probab.*, 20(4):1902–1931, 1992.

[14] A.S.Ustunel and M.Zakai. *Transformation of Measure on Wiener Space*. Springer-Verlag, Berlin, Heidelberg, New-York, 2000.

[15] A.V.Skorokhod. *Integration in Hilbert Space*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 79. - Springer-Verlag. New York; Heidelberg, 1974.

[16] A.V.Skorokhod. One generalization of the stochastic integral. *Theory Probab. Appl.*, 20(2):223–237, 1975.

[17] A.Yu.Shevliakov. Stochastic calculus with anticipating integrands. *Theory Probab. Math. Statist.*, 22(11):163–174, 1981.

[18] C.A.Tudor. Stochastic calculus with anticipating integrands. *Bernoulli*, 10(2):313–325, 2004.

[19] E.Pardoux and D.Nualart. Stochastic calculus with anticipating integrands. *Probab. Theory Related Fields*, 78:535–581, 1988.

[20] E.Pardoux and P. Protter. A two-sided stochastic integral and its calculus. *Probab. Theory Related Fields*, 78:15–19, 1987.

[21] A.M. Gomilko and A.A. Dorogovtsev. Localization of the extended stochastic integral. *Sbornik: Mathematics*, 197(9):1273–1295, 2006.

[22] M.Hitsuda. Formula for brownian partial derivatives. In *The Second Japan-USSR Symp. on Probab. Theory, Tbilisi; Springer-Verlag, Berlin, New York*, pages 111–114, 1972.

[23] M.Jolis and M. Sanz-Sole. Integrator properties of the skorokhod integral. *Stoch. and Stoch. Reports*, 41(3):163–176, 1992.

[24] N.N.Norin. Extended stochastic integral for non-gaussian measures in the locally-convex space. *Russian Math.Surveys*, 41(3):199–200, 1986.

[25] D. Nualart. *The Malliavin calculus and related topics*. Springer-Verlag, New York, 1995.

[26] O.Enchev. Stochastic integration with respect to gaussian random measures. In *Ph.D. Thesis, Sofia Univ., Sofia*, pages 52–60, 1983.

[27] Shigeyoshi Ogawa. Quelques propriétés de l'intégrale stochastique du type noncausal. *Japan J. Appl. Math.*, 1(2):405–416, 1984.

[28] O.G.Smolyanov. Differentiable measures on the group of functions taking values in a compact lee group. In *Abstract of the Sixth Intern. Vilnius Conf. on Probab. and Math. Statist., Vilnius*, pages 139–140, 1993.

[29] R.Buckhdan. *Quasilinear partial stochastic differential equations with out nonanticipation requirement*. Prepr. Humbolt Univ., No. 176, Berlin, 1989.

[30] R.Buckhdan, P. Malliavin, and D.Nualart. Multidimensional linear stochastic differential equations in the skorokhod sence. *Stoch. and Stoch. Reports*, 62(1–2):117–145, 1997.

[31] S.Tindel, C.A.Tudor, and F. Viens. Stochastic evolution equations with fractional brownian motion. *Probab. Theory Related Fields*, 127(2):186–Ű204, 2003.

[32] B. Symon. *The* $P(\varphi)_2$ *Euclidian (quantum) field theory*. Princeton Univ. Press., 1974.

[33] V.I.Klyackin. *Dynamics of stochastic systems*. Phizmathlit, Moscow, 2002.

[34] V.V.Baklan. One generalization of stochastic integral. *Dopovidi AN Ukraine, Ser.A*, 41(4):291–294, 1976.

[35] S. Watanabe. *Stochastic differential equations and Malliavin calculus*. Tata Inst, of Pundam. Research, Bombay, 1984.

[36] Yu.L.Dalecky and G.Ya.Sohadze. Absolute continuity of smooth measures. *Funct. Anal, and Appl.*, 22(2):77–78, 1988.

[37] Yu.L.Dalecky and S.N.Paramonova. One formula from gaussian measures theory and estimation of stochastic integrals. *Theory Probab. Appl.*, 19(4):845–849, 1974.

[38] Yu.L.Dalecky and S.V.Fomin. *Measures and differential equations in infinite-dimensional space*. Kluwer Acad. Publ. Boston, 1983.

[39] Yu.L.Dalecky and V.R.Steblovskaya. Smooth measures: absolute continuity, stochastic integrals, variational problems. In *Proc. of the Sixth USSR–Japan Symp. on Probab. Theory and Math. Statist., Kiev. - WSPC*, pages 52–60, 1991.