A Model in Population Genetics

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This research has been done as part of the group of A. Etheridge[1].

Introduction

One influential model in Population Genetics is the Fleming-Viot model trying to capture the evolution of the proportion of a specific allele in a population with event based reproduction. The total population size assumed to be constant, the proportion $(X_t)_{t\geq 0}$ of a subpopulation is modelled through the Markov Process with generator

$$A^u f(x) = x\Big(f((1-u)x+u) - f(x)\Big) + (1-x)\Big(f((1-u)x) - f(x)\Big),$$

where u plays the role of the impact of a single reproductive event. Taking the impact $u \downarrow 0$ in a diffusive time scale, one recovers the Wright-Fisher diffusion. This leads to the Λ-Fleming-Viot model modelling multi-scale reproduction events through the generator

$$Af = \int_0^1 A^u f \, \frac{\Lambda()}{u^2}$$

for some finite measure Λ on $(0,1)$.

In this talk, we will consider an extension of this model to general offspring distributions. More precisely, if ξ_u is a positive random variable with mean u, we consider the Markov Process (N,X) with generator

$$\begin{aligned} B^{u,\xi_u} f(n,x) = \frac{x}{n} \int_0^{+\infty} & f((1-u)n+\xi, (1-u)x+\xi) - f(n,x) P_{\xi_u}(\xi) \\ & + \left(1 - \frac{x}{n}\right) \int_0^{+\infty} f((1-u)n+\xi, (1-u)x) - f(n,x) P_{\xi_u}(\xi). \end{aligned}$$

In the generalised setting, we need to keep track of the total population size N as it not constant anymore.

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Some natural question are:

1. At which scale is there a limiting process in the small impact regime $u \downarrow 0$?
2. Can a generalised Λ-Fleming-Viot model be defined?
3. Can we characterise the ancestry in this model?
4. Is it possible to mimic the construction of the Spatial Λ-Fleming-Viot model to include space in the generalised model?

The two first problems will be subject of this talk. The project is ongoing: at the moment, we investigate the third question. The fourth problem is much more challenging than the nonspatial case. It turns out that the understanding of ancestry may be crucial to prove essential properties in the spatial case.

Main Results

In the following, denote by $(\xi_u)_{u\in(0,1)}$ a family of $[0,+\infty)$-valued random variables with $\mathbb{E}[\xi_u] = u$ and write P_u for their respective laws.

Write (N^u, X^u) for the process with generator B^{u,ξ_u}.

Theorem 1 (Convergence in the Small Impact Regime)**.** *Suppose that there exist $\gamma \in [0,1]$ and some jump measure Ξ on $(0,+\infty)$ such that*

$$\lim_{u\downarrow 0} \frac{1}{u}\mathbb{E}[\xi_u] = \gamma \quad \text{and} \quad \lim_{u\downarrow 0} \mathbb{E}[f(\xi_u)] = \int_0^{+\infty} f(\xi)\Xi(\xi)$$

for all continuous bounded functions $f : \mathbb{R} \to \mathbb{R}$ vanishing in a neighbourhood of zero. Then, for suitable initial conditions, the processes (N^u, X^u) converges in law in $\mathbb{D}_{[0,T]}(\mathbb{R}^2)$ to the Markov Process (N, X) started from (N_0, X_0) with generator

$$\begin{aligned} B^{\gamma,\Xi} f(N,X) &= (\gamma - N)\partial_N f(N,X) - \frac{X}{N}(\gamma - N)\partial_X f(N,X) \\ &\quad + \frac{X}{N}\int_0^{+\infty} f(N+\xi, X+\xi) - f(N,X)\Xi(\xi) \\ &\quad + \left(1 - \frac{X}{N}\right)\int_0^{+\infty} f(N+\xi, X) - f(N,X)\Xi(\xi). \end{aligned}$$

In particular, N is a Lévy-driven Ornstein-Uhlenbeck process.

Corollary 2 (Existence of the model)**.** *Let Λ be a finite measure on $(0,1)$. Suppose that the ξ_u satisfy the condition from Theorem 1. Then, for any $N_0 > 0$ and $0 \le X_0 \le N_0$, there is a unique Markov process $((N_t, X_t))_{t\ge 0}$ with generator*

$$Bf = \int_0^1 B^{u,\xi_u} f \, \frac{\Lambda()}{u}$$

started from (N_0, X_0).

References

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