Stochastic calculus for randomly scaled Gaussian processes related to generalized time-fractional evolution equations

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In [3] and [2] C. Bender developed the notion of an integral with respect to the non semi-martingale of fractional Brownian motion B_t^H for Hurst-parameter $0 < H < 1$, using an S-transform approach. An important result of [2] was the proof of an Itô-formula for this fractional integral. Building from the master thesis [4] of M. Gomolluch at the TU-Braunschweig, we generalize the approach of [2] to define an integral with respect to randomly scaled fractional Brownian motion, i.e. with respect to $X_t = \sqrt{A}B_t^H$ for a positive and independent random variable A with some regularity. In particular, we show that a direct generalisation of the Itô-formula of [2] continues to hold in this more general case. In the second part we use the generalized Itô-formula, together with techniques used in [1] by C. Bender and Y. Butko, to give a new proof of the know fact that a randomly scaled fractional Brownian motion X_t gives rise to solution of the generalized heat equation

$$u(t,x) = u_0(x) + \int_0^t k(t,s) \cdot \frac{1}{2}\Delta u(s,x) ds$$

via $u(t,x) = \mathbb{E}[u_0(x + X_t)]$ for certain homogeneous kernels k. Finally, we investigate the connection between integrals of continuous functions with respect to X_t and solutions to heat equations with time changed kernels $k(\sigma(t), \sigma(s)) \cdot \sigma'(s)$.

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2. C. Bender. An S-transform approach to integration with respect to a fractional Brownian motion. Bernoulli, 2003, 9, 6, 955–983.
3. C. Bender. An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. Stochastic Processes Appl., 2003, 104, 1, 81–106.
4. M. Gomolluch. Stochastisches Integral bezuglich verallgemeinerter grauer Brownscher Bewegung. MA thesis, Technische Universitât Braunschweig, 2022.

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