On Limit Theorems for Functional Autoregressive Processes with Random Coefficients

Sadillo Sharipov[1]

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space and $\mathbf{B}$ be a separable Banach space with norm $\|\cdot\|_{\mathbf{B}}$ and let $\mathbf{B}^*$ denote its dual space. For any real $p \geq 1$ denote $\mathbf{L}_{\mathbf{B}}^p$ the space of $\mathbf{B}$-valued random variables such that $\|X\|_{\mathbf{L}_{\mathbf{B}}^p}^p = \mathbf{E}\|X\|_{\mathbf{B}}^p$ is finite. We denote by $\mathcal{L}(\mathbf{B})$ the space of bounded linear operators over $\mathbf{B}$ equipped with usual uniform norm.

Let $(\rho_n, n \in \mathbb{Z})$ be a sequence of measurable random operators defined on $(\Omega, \mathcal{A}, \mathbf{P})$ with values in $\mathcal{L}(\mathbf{B})$ endowed with its Borel σ-field.

The class of Functional Autoregressive Processes with deterministic operator is a very flexible modeling and predictive tool for continuous time random process. The general theory of this model was presented in the pioneering work of Bosq [1] where developed estimation of its second order structure and derived an asymptotic theory. In this paper, we consider autoregressive processes with random coefficients with values in Banach spaces.

We first define a Banach-valued white noise.

Definition 1. A sequence $(\varepsilon_n, n \in \mathbb{Z})$ of $\mathbf{B}$-valued random variables is called a strong white noise (innovation process) if it is an i.i.d. and such that $\mathbf{E}\varepsilon_0 = 0$ and $\mathbf{E}\|\varepsilon_0\|_{\mathbf{B}}^2 < \infty$.

Now recall the definition of Banach-valued random coefficients autoregressive processes of order 1.

Definition 2. A sequence $(X_n, n \in \mathbb{Z})$ of $\mathbf{B}$-valued random variables satisfying the following recursion equation

$$X_n - \mu = \rho_n(X_{n-1} - \mu) + \varepsilon_n, \quad n \in \mathbb{Z} \tag{1}$$

where $\mu \in \mathbf{B}$, is called a Banach-valued random coefficients autoregressive process of order 1 (we abbreviate by BRCA(1)).

The sequence (1) defines a nonlinear functional time series model, thus extends the class of functional autoregressive models paramount in linear functional time series analysis (see [4]). The process defined by (1) is used to handle possible nonlinear features of data and includes nonlinear models, threshold models and some double stochastic time series.

We recall the notion of $r-$smooth Banach spaces, $1 < r \leq 2$.

Definition 3. We say that a separable Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ is $r-$smooth $(1 < r \leq 2)$ if there exists an equivalent norm $\|\cdot\|'$ such that

$$\sup_{t>0}\{\frac{1}{t^r}\sup\{||x+ty||' + ||x-ty||' - 2 : ||x||' = ||y||' = 1\}\} < \infty.$$

This notion was introduced by Pisier [3]. Note that if $\mathbf{B}$ is $r-$smooth then it is $r'-$smooth for any $r' < r$. According to [1], if $\mathbf{B}$ is $r-$smooth and separable then there exist a constant $D \geq 1$ such that, for any sequence of $\mathbf{B}-$valued martingale difference $(\xi_i, i \geq 1)$

$$\mathbf{E}||\xi_1 + ... + \xi_n||_{\mathbf{B}}^r \leq D \sum_{i=1}^{n} \mathbf{E}||\xi_i||_{\mathbf{B}}^r.$$

Therefore, we may claim that these spaces play the same role with respect to martingales as spaces to type p do with respect to the sums of independent variables.

[1]V.I. Romanovskiy Institute of Mathematics, Tashkent, Uzbekistan. Email: sadi.sharipov@yahoo.com

Our main purpose is to derive some limit theorems for Banach-valued autoregressive processes with random coefficients.

We consider estimation of the mean μ from observations $X_1, \dots, X_n$, where (X_n) is BRCA(1) process. The natural estimator is the sample mean

$$\overline{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Observe that $\mathbf{E}\overline{X}_n = \mu$, that is, that $\mathbf{E}\overline{X}_n$ is an unbiased estimator of μ. Throughout the paper, we set $S_n := X_1 + \cdots + X_n$, $n \geq 1$.

We now give our main results. We first study sufficient conditions for the existence and uniqueness of strictly stationary BRCA(1) process. In order to provide the existence and uniqueness of strictly stationary solution of BRCA(1) we need the following conditions:

(C1(p)): The random variables $(\rho_n, n \in \mathbb{Z})$ are i.i.d. and belongs to $\mathbf{L}^p_{\mathbf{B}}$, $p \geq 1$.

(C2): The two sequences $(\rho_n, n \in \mathbb{Z})$ and $(\varepsilon_n, n \in \mathbb{Z})$ are independent.

(C3(p)): $\mathbf{E}||\rho_0||^p_{\mathcal{L}} < 1$, $p \geq 1$.

Lemma 1. Let $(\varepsilon_n, n \in \mathbb{Z})$ be an i.i.d. sequence of $\mathbf{B}-$valued random variables belongs to $\mathbf{L}^p_{\mathbf{B}}$, $p \geq 1$. Assume the conditions $(C1(p)) - (C3(p))$ hold. Then the equation (1) has a unique strictly stationary solution given by

$$X_n = \mu + \sum_{j=0}^{\infty} A_{n,j}\varepsilon_{n-j}, \quad n \in \mathbb{Z}, \tag{2}$$

where $A_{n,0} = I$ and $A_{n,j} = \rho_n * ... * \rho_{n-j+1}$ for $j \geq 1$, the series on the right-hand side of (2) converges absolutely almost surely and in $\mathbf{L}^p_{\mathbf{B}}$, $p \geq 1$.

The next result concerns consistency of the sample mean $\overline{X}_n$.

Theorem 1. Let $(X_n, n \in \mathbb{Z})$ be BRCA(1) process defined in $\mathbf{L}^1_{\mathbf{B}}$ with $\mathbf{E}X_0 = 0$. If conditions $(C1(1)) - (C3(1))$ hold, then $\overline{X}_n$ converges to μ almost surely.

We now deal with CLT for BRCA(1) process.

Theorem 2. Let $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ be 2-smooth Banach space. Assume the conditions $(C1(2)) - (C3(2))$ hold for BRCA(1) process. Then

$$\sqrt{n}\left(\frac{S_n}{n} - \mu\right) \xrightarrow{\mathcal{D}} N \sim \mathcal{N}(0, \Gamma), \quad n \to \infty,$$

where $\Gamma = (I - \mathbf{E}(\rho_0))^{-1}(X_1 - \mathbf{E}(\rho_0)X_0)$.

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