

Article

Hitting Probabilities for a Class of Gaussian Integrators via Second Quantization

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Abstract

In our paper, we consider Gaussian processes in the form of $\eta(t) = \int_0^1 A(\mathbf{1}_{[0,t]})(s)dw(s)$, $t \in [0, 1]$, where $\{w(t); t \in [0, 1]\}$ is a standard Wiener process in $\mathbb{R}^d$ and A is a continuous linear operator on $L_2([0, 1])$ into itself. For the domain of $\mathcal{D} \subset \mathbb{R}^d$ with a C^1 smooth boundary, the following probability is considered for $x \in \mathcal{D}$ $P\{\exists \tau \in [0, 1] : x + \eta(\tau) \in \partial\mathcal{D}\}$. It is well known that, for the Wiener process, such hitting probabilities satisfy the parabolic boundary-value problem for heat equations. Note that, in general, the process (η) may be non-Markovian and may fail to be a semimartingale. Despite this, η can be considered an application of a second quantization operator to w . Our main contribution is the derivation of a representation for hitting probabilities as a series of Wiener–Itô multiple integrals with the kernel obtained from the Green function on the domain $\mathcal{D}$ and tensor powers of operator A .

Keywords: Wiener–Itô expansion; second quantization; Gaussian process**MSC:** 60H07; 81T40; 60G15

1. Introduction

1.1. Background

Hitting times and hitting probabilities are fundamental objects of study in the theory of stochastic processes, with far-reaching applications across financial mathematics, risk theory, neuroscience, and engineering reliability analysis. Owing to their favorable mathematical properties and rich application scenarios, the hitting problem for Gaussian processes has long been a central research topic in stochastic analysis [1]. For Gaussian processes possessing the Markov property (most notably, standard Brownian motion), the hitting problem admits a complete and well-developed classical theory. Using the reflection principle and partial differential equation methods, exact distributions of hitting times for both one-dimensional and multidimensional Brownian motion have been rigorously derived, establishing a deep and fundamental connection between stochastic hitting problems and classical analysis [1,2]. However, for general, non-Markovian Gaussian processes, the hitting problem remains notoriously difficult. Non-Markovianity implies that the future evolution of the process depends not only on its current state but also on its entire historical trajectory, rendering classical Markovian techniques—such as the strong Markov property, reflection principle, and PDE methods—directly inapplicable [2]. Nevertheless, substantial progress has been made in the study of hitting problems for non-Markovian Gaussian



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processes over the past few decades. Early work primarily focused on fractional Brownian motion [3], while in recent years, researchers have developed powerful new approaches, including Wiener–Itô expansion [4] and second quantization operator theory [5], extending the scope of investigation to a much wider class of Gaussian processes beyond fractional Brownian motion. Despite these significant advances, several critical limitations persist in the existing literature: most methods exhibit limited generality, there is an absence of a unified theoretical framework, and a notable gap remains in the study of operator-perturbed Gaussian processes. In particular, the hitting properties of Gaussian processes obtained by applying linear operators to Brownian motion have not been thoroughly investigated [6]. In this paper, we address the hitting-time and hitting-probability problems for a distinguished class of Gaussian processes called integrators. Introduced by Dorogovtsev [7] in the context of stochastic integration theory and anticipating stochastic differential equations, integrators form a rich class that includes Brownian motion, Brownian bridges [8], the Ornstein–Uhlenbeck process [9], and many other processes widely used in applications. It was discovered that integrators can be used to model linear polymers [7]. Beyond their theoretical importance, key properties such as their local self-intersection times [10] and corresponding non-normalized versions have been extensively studied. To clarify the mathematical regularities of the object under study, we recall that a generalized Gaussian integrator $(\eta(t))$ on $[0, 1]$ in the sense of Dorogovtsev [7] is defined via the action of a continuous linear operator (A) on $L_2([0, 1]; \mathbb{R}^d)$, driven by a standard d -dimensional Wiener process $(w(t))$, which can be formally represented as

$$\eta(t) = \int_0^1 A(\mathbf{1}_{[0,t]})(s) dw(s), \quad t \in [0, 1], \quad (1)$$

where A satisfies the structural orthogonal decomposition $(A^*A + R^*R = I)$ with a finite-rank perturbation operator, i.e., R ($\|R\| \leq 1$).

The intrinsic connection between such operator-induced integrators and the second quantization framework lies in the transformation of underlying Gaussian measures. Specifically, the linear deformation of the covariance structure induced by A can be lifted to the entire Wiener chaos space. The Radon–Nikodym derivative of the perturbed law with respect to the standard Wiener measure is precisely given by the exponential martingale, whose algebraic structure is perfectly captured by the second quantization operator $(\Gamma(R))$ acting on the Hilbert–Schmidt Fock space. This algebraic correspondence allows us to map the geometric hitting event into an analytic operator-theoretic evaluation, bypassing the lack of Markovianity.

1.2. Contributions of This Paper

This paper studies hitting probabilities for a class of non-Markovian Gaussian processes generated by continuous linear operators. Positioned against classical and state-of-the-art literature, our main contributions are threefold and can be summarized as follows:

- **A General Framework for Non-Markovian Gaussian Integrators:** First, we broaden the class of processes for which explicit hitting-probability results are available. Moving beyond classical Markovian diffusions and standard Gaussian processes, we consider multidimensional Gaussian integrators of the following form:

$$\eta(t) = \int_0^1 A(\mathbf{1}_{[0,t]})(s) dw(s), \quad t \in [0, 1],$$

where A is a continuous linear operator on $L_2([0, 1])$. This family exhibits genuinely non-Markovian temporal dependence and covers cases falling outside the scope of classical Itô calculus (e.g., various generalized bridges and Ornstein–Uhlenbeck

processes). For a fixed domain ($\mathcal{D} \subset \mathbb{R}^d$) with a smooth boundary and $x \in \mathcal{D}$, we systematically analyze the hitting probability ($\mathbb{P}\{\exists \tau \in [0, 1] : x + \eta(\tau) \in \partial\mathcal{D}\}$). Relative to the foundational framework established by Dorogovtsev [5], we extend the analysis to a more general operator-induced class.

- **A Second Quantization Operator-Theoretic Methodology:** Second, we develop a novel methodology that bridges partial differential equations and infinite-dimensional analysis. Traditional approaches to hitting probabilities predominantly rely on local PDE techniques, martingale theory, or extreme value theory. In contrast, we introduce the indicator functional ($W_{x,s}$) for the hitting event and demonstrate that its Fourier–Wiener transform ($T(W_{x,t})(h)$) solves a parabolic boundary-value problem with a generator ($\frac{1}{2}\Delta + h(t) \cdot \nabla$). Utilizing Duhamel’s principle, we identify the Wiener–Itô chaos kernels (g_n^x) for Brownian motion in terms of iterated integrals involving the heat equation’s Green function. By introducing second quantization operators ($\Gamma(R)$) for operators with $\|R\| < 1$, we transfer this Brownian chaos expansion to the general integrator (η). This approach reduces the problem to evaluation of a second quantization operator, offering a clear algebraic structure suited for perturbation analysis.
- **Explicit Series Representation and Continuity Justification:** Third, under the finite-rank perturbation assumption, we obtain an explicit and computable series representation for the hitting probability. Unlike most existing results that offer only asymptotic estimates or two-sided bounds, our representation takes a closed-form series formulated in terms of transformed kernels and Hermite polynomials evaluated at the origin. We rigorously establish continuity results for both the hitting functionals and the Wiener–Itô series under second quantization, which justifies evaluations of the resulting expressions at appropriate Gaussian vectors taking values of zero. Finally, benchmarked against recent advances such as the chaos decomposition of Gaussian nodal volumes [4], our result yields the first explicit series formula for hitting probabilities strictly within the second quantization framework.

2. Second Quantization for Functionals from Brownian Motion in a Tube

2.1. Notation

To make the paper accessible to a broader audience, we briefly collect the main notational conventions and definitions of Wiener space analysis used throughout this work.

- **Tensor and Scalar Products:** $H^{\otimes n}$ denotes the n -fold tensor product of a Hilbert space (H ; typically, $L_2([0, 1])$ or $\mathbb{R}^d$). It is equipped with the canonical inner product ($\langle \cdot, \cdot \rangle_{H^{\otimes n}}$), which, for simple tensors, satisfies

$$\langle h_1 \otimes \cdots \otimes h_n, g_1 \otimes \cdots \otimes g_n \rangle_{H^{\otimes n}} = \prod_{i=1}^n \langle h_i, g_i \rangle_H.$$

- **Second Quantization Operator.** Given a bounded linear operator (R) on H with $\|R\| \leq 1$, the second quantization operator ($\Gamma(R)$) [11] is a bounded linear operator acting on the symmetric Fock space ($\Gamma(H) = \bigoplus_{n=0}^{\infty} H^{\odot n}$ where $H^{\odot n}$ is the n -fold symmetric tensor power of H). It is defined by its restriction to the n -th chaos:

$$\Gamma(R)|_{H^{\odot n}} = R^{\otimes n}.$$

- **Wick Products:** For $h \in H$, the Wick exponential is defined as

$$:\exp(h): = \sum_{n=0}^{\infty} \frac{h^{\otimes n}}{n!} \in \Gamma(H).$$

The Wick product [12], denoted by $*$, is the unique continuous bilinear operation on the Fock space such that $:\exp(h) : * :\exp(g) := :\exp(h+g) :$ for all $h, g \in H$.

- **Ogawa Symmetric Integral:** The Ogawa symmetric integral [13] extends stochastic integration to anticipation of integrands with respect to a Gaussian integrator (η) . The Ogawa symmetric integral of a suitable integrand (f) is denoted as $\int_0^1 f(t) \circ d\eta(t)$ using circular notation. More generally, the n -fold multiple Ogawa symmetric integral of a kernel (b_n) is defined via the continuous extension of the integral of simple functions via Wiener–Itô chaos expansion, which is formally denoted by

$$\int_0^1 \cdots \int_0^1 b_n(t_1, \dots, t_n) d\eta(t_1) \circ \cdots \circ d\eta(t_n).$$

For detailed expositions, we refer the reader to [7,14].

2.2. Basic Definition and Main Idea

Initially, the integrator was defined as follows.

Definition 1. [7] A centered Gaussian process $(\{\eta(t), t \in [0, 1]\})$ is called an integrator if there exists a positive constant $(C > 0)$ such that for an arbitrary $n \geq 1$, a partition of $0 = t_0 < \cdots < t_n = 1$ and real numbers $a_1, \dots, a_n$,

$$\mathbb{E} \left[\sum_{k=0}^{n-1} a_k (\eta(t_{k+1}) - \eta(t_k)) \right]^2 \leq C \sum_{k=0}^{n-1} a_k^2 (\Delta t_k)^2, \quad (2)$$

where $\Delta t_k := t_{k+1} - t_k$.

This condition (2) means that for every function $(f \in L_2([0, 1]))$, the integral, i.e.,

$$\int_0^1 f(t) d\eta(t),$$

is well-defined as a limit in the square mean of the step functions. Furthermore, setting $n = 1$ in the defining inequality yields $\mathbb{E}[(\eta(t) - \eta(s))^2] \leq C(t - s)^2$ for any $s, t \in [0, 1]$. Since η is Gaussian, according to Kolmogorov's continuity theorem, the integrator (η) is guaranteed to admit a sample-continuous modification. Additionally, the aforementioned mean-square continuity immediately ensures that η possesses a jointly continuous covariance function $(K(s, t) = \mathbb{E}[\eta(s)\eta(t)])$ on $[0, 1] \times [0, 1]$. Finally, there is a representation of integrators through the Gaussian white noise in Hilbert space, which will be extensively used later.

Lemma 1. [7] Let $\{\eta(t), t \in [0, 1]\}$ be an integrator. Then, there exist a Wiener process $(\{w(t), t \in [0, 1]\})$ and a continuous linear operator (A) on the $L_2([0, 1])$ space into itself such that

$$\eta(t) = \int_0^1 A(\mathbf{1}_{[0,t]})(s) dw(s).$$

To avoid any ambiguity regarding the dimensionality and the action of the operator, we emphasize that A operates strictly on the time domain. Let $w(t) = (w_1(t), \dots, w_d(t))^T$ be a d -dimensional Wiener process, with the stochastic integral defining $\eta(t)$ understood in the component-wise Itô sense and the same scalar integrand $(A(\mathbf{1}_{[0,t]})(s))$ integrated against each coordinate process $(w_i(s))$ separately, yielding the i -th coordinate of $\eta(t)$

$$\eta_i(t) = \int_0^1 A(\mathbf{1}_{[0,t]})(s) dw_i(s), \quad i = 1, 2, \dots, d.$$

Thus, $\eta(t) = (\eta_1(t), \dots, \eta_d(t))^T$ is a well-defined d -dimensional Gaussian process.

To establish a rigorous foundation for our hitting-time analysis, we systematically elaborate on the core covariance structure, the sample-path regularity of the multidimensional Gaussian integrator $(\eta(t))$.

By construction, $\eta(t) = (\eta_1(t), \dots, \eta_d(t))^T$ is a centered, multi-dimensional Gaussian process. Since the component-wise stochastic integration is governed by the continuous linear operator (A) acting on the time-domain space $(L_2([0, 1]))$, the covariance structure of the process is fully determined by the geometry of A . Specifically, for each coordinate $(i = 1, \dots, d)$, the auto-covariance function $(K(s, t))$ for any $s, t \in [0, 1]$ is given by

$$K(s, t) := \mathbb{E}[\eta_i(s)\eta_i(t)] = \left\langle A\mathbf{1}_{[0,s]}, A\mathbf{1}_{[0,t]} \right\rangle_{L_2([0,1])}. \quad (3)$$

Since the driving scalar Wiener processes (w_i) and (w_j) are independent for $i \neq j$, we have $\mathbb{E}[\eta_i(s)\eta_j(t)] = 0$. Consequently, the full covariance matrix of the vector process is compactly represented as $K(s, t) \cdot I_d$, where I_d is the $d \times d$ identity matrix.

A defining feature of the Gaussian integrator $(\eta(t))$ is that it is generically non-Markovian. The integral operator (A) typically introduces non-local memory effects across the time horizon, implying that the future evolution of the sample paths depends heavily on their entire historical trajectory rather than solely on the current state.

Consider a case when $\|A\| < 1$. We define

$$R = (I - A^*A)^{\frac{1}{2}},$$

$$\zeta(t) = \int_0^1 R(\mathbf{1}_{[0,t]})(s)dw'(s), \quad t \in [0, 1],$$

where w' is a Wiener process independent of w . It is straightforward to verify that the sum of

$$\tilde{w}(t) = \eta(t) + \zeta(t), \quad t \in [0, 1],$$

is a Wiener process. Suppose that $\Phi : C([0, 1]) \rightarrow \mathbb{R}$ is a bounded, measurable function. Under the structural assumption that R is a finite-rank operator of rank k , the image of R is spanned by an orthonormal set of functions $(\{e_1, \dots, e_k\})$ in $L_2([0, 1])$. Consequently, the process (ζ) can be explicitly represented as

$$\zeta(t) = \sum_{i=1}^k \xi_i \int_0^t e_i(s)ds, \quad t \in [0, 1],$$

where $\vec{\xi} = (\xi_1, \dots, \xi_k)^T$ is a standard, k -dimensional Gaussian vector. Therefore, the condition of $\{\zeta = 0\}$ in the path space is strictly equivalent to the finite-dimensional event $(\{\vec{\xi} = \mathbf{0}\})$ in $\mathbb{R}^k$. Since the Gaussian vector $(\vec{\xi})$ possesses a smooth and strictly positive density with respect to the Lebesgue measure on $\mathbb{R}^k$, the conditioning on this zero-measure event is rigorously well-defined via the standard regular conditional probability distribution. Evaluated on the Polish space $(C([0, 1]))$ equipped with the uniform topology, we rigorously obtain the identity as

$$\mathbb{E}[\Phi(\eta)] = \mathbb{E}[\Phi(\tilde{w}) \mid \zeta = 0]. \quad (4)$$

Note that the conditional expectation with respect to ζ can be calculated as a second quantization operator acting on functionals of $\tilde{w}$. Namely, since ζ is defined as

$$\zeta(t) = \int_0^1 R(\mathbf{1}_{[0,t]})(s)dw'(s), \quad t \in [0, 1],$$

then for an arbitrary $h \in L_2([0, 1])$,

$$\int_0^1 h(s) d\tilde{w}(s) = \int_0^1 (Ah)(s) dw(s) + \int_0^1 (Rh)(s) dw'(s),$$

and

$$\int_0^1 (Rh)(s) dw'(s) = \int_0^1 h(s) d\zeta(s)$$

is measurable with respect to ζ . Consequently, the conditional expectation (4) is given by

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ \int_0^1 h(s) d\tilde{w}(s) - \frac{1}{2} \int_0^1 h^2(s) ds \right\} \middle| \zeta \right] \\ &= \exp \left\{ \int_0^1 (Rh)(s) dw'(s) - \frac{1}{2} \int_0^1 (Rh)^2(s) ds \right\} \\ &= \Gamma(R) \exp \left\{ \int_0^1 h(s) dw'(s) - \frac{1}{2} \int_0^1 h^2(s) ds \right\}. \end{aligned}$$

Here $\Gamma(R)$ is the second quantization operator acting on the space of square-integrable functionals of w' . To establish this relation with full mathematical rigor, we recall the formal definition and chaotic characterization of the second quantization operator from standard Malliavin calculus and constructive quantum field theory.

Proposition 1 (Second Quantization via Wiener–Itô Expansion [11]). *Let R be a bounded linear operator on $L_2([0, 1])$. The second quantization operator $(\Gamma(R))$ uniquely extends to a bounded linear operator on the space of square-integrable functionals $(L_2(\Omega_{w'}))$. Specifically, let $\alpha \in L_2(\Omega_{w'})$ be a functional admitting the canonical Wiener–Itô chaos expansion:*

$$\alpha = \mathbb{E}[\alpha] + \sum_{n=1}^{\infty} \int_{\Delta_n} a_n(t_1, \dots, t_n) dw'(t_1) \cdots dw'(t_n),$$

where $\Delta_n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid 0 \leq t_1 \leq \dots \leq t_n \leq 1\}$. Then, the action of $\Gamma(R)$ on α is given explicitly by

$$\Gamma(R)\alpha = \mathbb{E}[\alpha] + \sum_{n=1}^{\infty} \int_{\Delta_n} (R^{\otimes n} a_n)(t_1, \dots, t_n) dw'(t_1) \cdots dw'(t_n), \quad (5)$$

where $R^{\otimes n}$ denotes the n -fold tensor power of the operator (R) .

Calculating (5) when $\zeta = 0$ is equivalent to the evaluation at $w' = 0$, suppose that the $b_n \in L_2([0, 1]^n)$ kernel is symmetric. To find a concrete representation of this integral, we utilize a spatial basis expansion justified by the underlying Hilbert-space isometry.

Let $\{e_k; k \geq 1\}$ be an orthonormal basis in $L_2([0, 1])$. According to Parseval's identity on the tensor-product Hilbert space $(L_2([0, 1]^n))$, the symmetric kernel (b_n) can be expanded via the orthonormal basis $(\{e_{k_1} \otimes \dots \otimes e_{k_n}\})$ as follows:

$$b_n(t_1, \dots, t_n) = \sum_{k_1, \dots, k_n=1}^{\infty} b_{k_1, \dots, k_n} e_{k_1}(t_1) \cdots e_{k_n}(t_n),$$

where the expansion coefficients are given by

$$b_{k_1, \dots, k_n} = \int_0^1 \cdots \int_0^1 b_n(t_1, \dots, t_n) e_{k_1}(t_1) \cdots e_{k_n}(t_n) dt_1 \cdots dt_n, \quad k_1, \dots, k_n \geq 1.$$

Next, we map this deterministic expansion into the stochastic framework. Let $\xi_k = \int_0^1 e_k(s)dw'(s)$ for $k \geq 1$ denote the independent standard Gaussian coordinates. By virtue of the chaos decomposition theorem, there exists an isometric isomorphism between $L_2([0, 1]^n)$ and the n -th Wiener chaos. This fundamental theorem guarantees that the L_2 -convergent series of the kernel can be integrated term by term. Consequently, the multiple Wiener–Itô integral can be rigorously expressed via the Wick products of the Gaussian coordinates:

$$\begin{aligned} \int_{\Delta_n} b_n(t_1, \dots, t_n) dw'(t_1) \cdots dw'(t_n) &= \frac{1}{n!} \int_0^1 \cdots \int_0^1 b_n(t_1, \dots, t_n) dw'(t_1) \cdots dw'(t_n) \\ &= \frac{1}{n!} \sum_{k_1, \dots, k_n=1}^{\infty} b_{k_1, \dots, k_n} \xi_{k_1} * \cdots * \xi_{k_n}. \end{aligned}$$

Here, the multiple integral, i.e.,

$$\int_0^1 \cdots \int_0^1 b_n(t_1, \dots, t_n) dw'(t_1) \cdots dw'(t_n)$$

is defined in the Skorokhod sense [15], and

$$\xi_{k_1} * \cdots * \xi_{k_n}$$

is the Wick product [16] of $\xi_{k_1}, \dots, \xi_{k_n}$. Such a product is obtained from the usual product of $\xi_{k_1}, \dots, \xi_{k_n}$ by collecting identical multipliers and substituting the corresponding powers with Hermite polynomials of the same degree. For example,

$$\xi_1 * \xi_2 * \xi_1 = H_2(\xi_1) \cdot \xi_2 = (\xi_1^2 - 1)\xi_2.$$

Hence, the value is

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 b_n(t_1, \dots, t_n) dw'(t_1) \cdots dw'(t_n) |_{w'=0} \\ &= \sum_{k_1, \dots, k_n=1}^{\infty} b_{k_1, \dots, k_n} \xi_{k_1} * \cdots * \xi_{k_n} |_{\xi_1=\xi_2=\dots=0}. \end{aligned}$$

Using the generating function of the classical Hermite polynomials, i.e.,

$$\exp\left(ty - \frac{1}{2}t^2\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(y), \quad (6)$$

setting the spatial variable to $y = 0$ yields the explicit evaluation at the origin for any $n \geq 0$:

$$H_{2n+1}(0) = 0, \quad H_{2n}(0) = (-1)^n (2n-1)!! = (-1)^n \frac{(2n)!}{n!2^n}.$$

To establish a unified formulation connecting these boundary evaluations with the underlying Gaussian probability space, let $\xi \sim \mathcal{N}(0, 1)$ be a standard Gaussian random variable. By comparing the Taylor expansion of its characteristic function ($\mathbb{E}[e^{it\xi}] = \exp(-\frac{1}{2}t^2)$) with (6) at $y = 0$, we obtain the following fundamental identity:

$$H_n(0) = i^n \mathbb{E}[\xi^n], \quad n \geq 0, \quad (7)$$

where $i^2 = -1$. More generally, let $\xi = (\xi_1, \dots, \xi_m)^T \sim \mathcal{N}(\mathbf{0}, I_m)$ be a vector of independent standard Gaussian variables. For any multi-index $(J = (j_1, \dots, j_m) \in \mathbb{N}^m)$ with a total order

of $|J| = j_1 + \cdots + j_m = n$, the product of the Hermite polynomials evaluated at the origin admits the following unified expectation representation:

$$H_{j_1}(0) \cdots H_{j_m}(0) = i^n \mathbb{E} \left[\xi_1^{j_1} \cdots \xi_m^{j_m} \right]. \quad (8)$$

When substituting (8) into the Wiener–Itô chaos expansion of the hitting functional to evaluate the probability at the origin, we encounter the problem of interchanging an infinite summation (limit) over the chaos order (n) with the expectation operator ($\mathbb{E}[\cdot]$). We now state the sufficient condition and provide the rigorous mathematical justification for this operation below.

Proposition 2. *Let $\|R\| < 1$. Then, the sufficient condition for the validity of interchanging the infinite summation and the expectation operator is satisfied, i.e.,*

$$\sum_{n=0}^{\infty} \sum_{|J|=n} C_J \cdot \mathbb{E} \left[\xi_1^{j_1} \cdots \xi_m^{j_m} \right] = \mathbb{E} \left[\sum_{n=0}^{\infty} \sum_{|J|=n} C_J \xi_1^{j_1} \cdots \xi_m^{j_m} \right], \quad (9)$$

where C_J denotes the deterministic contraction coefficients containing the kernels and factorial weights.

Proof. To justify the limit-expectation interchange, we verify the conditions of the Lebesgue Dominated Convergence Theorem. Under the structural contraction assumption of $\|R\| < 1$, the eigenvalues satisfy $|\beta_k| \leq \|R\| < 1$. The absolute series inside the expectation is uniformly bounded on the compact set $\{|\xi_k| \leq 1, k = 1, \dots, m\}$ by

$$\sum_{n=0}^{\infty} \sum_{|J|=n} |C_J \xi_1^{j_1} \cdots \xi_m^{j_m}| \leq c^m (\mathbb{E} \alpha^2)^{\frac{1}{2}} \prod_{k=1}^m \frac{1}{1 - |\beta_k|} < +\infty.$$

Since the tail of the Gaussian joint density decays exponentially, the uniform convergence on compact sets, combined with the Cauchy–Schwarz inequality, ensures that the entire infinite sum is dominated by an $L_1(\mathbb{P})$ -integrable envelope function. Specifically,

$$\mathbb{E} \left[\sum_{n=0}^{\infty} \sum_{|J|=n} |C_J \xi_1^{j_1} \cdots \xi_m^{j_m}| \right] \leq c^m (\mathbb{E} \alpha^2)^{\frac{1}{2}} \prod_{k=1}^m \frac{1}{1 - |\beta_k|} < +\infty.$$

The absolute integrability guarantees that the series converges in $L_1(\Omega, \mathbb{P})$. Therefore, according to the Dominated Convergence Theorem, the interchange of the infinite summation limit and the expectation operation in (9) is rigorously justified. $\square$

Hence,

$$\begin{aligned} & \int_0^1 \cdots \int_0^1 b_n(t_1, \dots, t_n) dw'(t_1) \cdots dw'(t_n) |_{w'=0} \\ &= \mathbb{E} \left[\sum_{k_1, \dots, k_n=1}^{\infty} b_{k_1, \dots, k_n}(i\xi_{k_1}) \cdots (i\xi_{k_n}) \right] \\ &= i^n \mathbb{E} \left[\int_0^1 \cdots \int_0^1 b_n(t_1, \dots, t_n) dw'(t_1) \circ dw'(t_2) \circ \cdots \circ dw'(t_n) \right]. \end{aligned}$$

On the right-hand side, we use the Ogawa symmetric integral [13]. In elementary cases, this integral coincides with the Stratonovich integral. Accordingly, our strategy for computing probabilities associated with the integrator (η) is outlined below.

Remark 1 (Summary of the Main Proof Strategy). *The theoretical transition from the baseline Brownian space to the integrator's probability space via second quantization can be systematically executed through the following operational roadmap:*

1. *Operator Deformation: We define the structural self-adjoint contraction operator (R) on $L_2([0, 1])$ via the following complementary orthogonal relation:*

$$R = \sqrt{I - A^*A},$$

2. *Chaotic Lifting: The second quantization operator ($\Gamma(R)$) acts on the baseline functional (Φ), mapping its chaotic components into a well-defined Wiener–Itô series expansion with respect to the auxiliary Brownian motion (w'):*

$$\Gamma(R)\Phi(w') = \sum_{n=0}^{\infty} \int_{\Delta_n} b_n(t_1, \dots, t_n) dw'(t_1) \cdots dw'(t_n).$$

3. *Expectation Representation: By projecting the lifted kernels onto the boundary (the origin) using the Gaussian moment identity, the final desired hitting probability ($\mathbb{E}[\Phi(\eta)]$) under the deformed integrator measure is explicitly captured by the following structured series:*

$$\mathbb{E}[\Phi(\eta)] = \mathbb{E} \left[\sum_{n=0}^{\infty} i^n \int_{\Delta_n} b_n(t_1, \dots, t_n) dw'(t_1) \circ dw'(t_2) \circ \cdots \circ dw'(t_n) \right].$$

Next, we evaluate the expectations of functionals of integrators. Let us suppose that the function expressed as

$$\Phi : C([0, 1]) \rightarrow \mathbb{R}$$

is bounded and measurable. Consider an integrator (x) that is generated by the operator (A) with a norm not greater than one. Operator $R = \sqrt{I - A^*A}$ is defined as before, and the integrator (y) is constructed based on R . This structural identification leads to the following functional representation for the hitting probability under the transformed measure:

$$\mathbb{E}[\Phi(x)] = \mathbb{E}[\Phi(x + y(w')) \mid y(w') = 0] = \Gamma(R)\Phi(y(w'))|_{w'=0}.$$

This pivotal equality is mathematically justified by the smooth regularity properties of the conditional expectation ($\mathbb{E}[\Phi(x + y(w')) \mid y(w')]$) and the second-quantized functional ($\Gamma(R)\Phi(y(w'))$). Within the continuous framework of generalized Dirichlet forms and infinite-dimensional Sobolev embeddings, these objects are rigorously verified to belong to the local domain of the generator, thereby ensuring that the pointwise evaluation at the origin Gaussian coordinate ($y(w') = 0$) is completely well-defined. The technical proofs for these underlying regularity estimates and the localized convergence are systematically established in Section 4. But now, one can check the simple example in which, instead of Wiener processes, we use standard Gaussian variables.

Example 1. *Consider the standard Gaussian variable (ξ). The Wiener–Itô expansion of square-integrable functionals of ξ is now expansion with respect to the orthogonal system of Hermite polynomials:*

$$\alpha = \sum_{n=0}^{\infty} a_n H_n(\xi).$$

Here,

$$H_n(u) = (-1)^n e^{\frac{u^2}{2}} \frac{d^n}{du^n} e^{-\frac{u^2}{2}}, \quad n \geq 0.$$

Since ξ is a random variable, instead of $L_2([0, 1])$, the one-dimensional space ($\mathbb{R}$) must be taken. Operators in this space are only multiplications on the constant. It can be checked that, for $|c| \leq 1$,

$$\Gamma(c)\alpha = \sum_{n=0}^{\infty} c^n a_n H_n(\xi).$$

Now, the analog of the integrator related to the operator of multiplication on c is simply $c\xi$. Let us find the expectation of $H_n(c\xi)$ using the proposed method. Consider a standard Gaussian variable (ξ') that is independent of ξ , and define $c' = \sqrt{1 - c^2}$. Then,

$$\mathbb{E}[H_n(c\xi + c'\xi')|\xi'] = (c')^n H_n(\xi'),$$

where both sides are continuous functions of ξ' . Hence,

$$\mathbb{E}[H_n(c\xi)|\xi'] = (c')^n H_n(0).$$

For $n = 2k + 1$,

$$H_n(0) = 0,$$

and for $n = 2k$,

$$\mathbb{E}[H_{2k}(c\xi)] = (2k - 1)!!(1 - c^2)^k.$$

We aim to extend this argument to the general infinite-dimensional setting. Of course, the existence of a continuous version of the studied functionals is the problem, which will be solved in certain special cases. Therefore, the primary objective of this paper is to determine the probability that a multi-dimensional integrator reaches the boundary of the $\mathcal{D}$ domain prior to time 1. To achieve this via the aforementioned framework, it is necessary to establish the continuity of the corresponding conditional probability and the second quantization operator. This continuity framework ultimately guarantees the existence of continuous versions, thereby establishing our main result.

3. Wiener–Itô Expansion of the Hitting Event

As described in the previous section, we are looking for the Wiener–Itô expansion of the following functional of the standard Wiener process (w) starting from a point ($x \in \mathcal{D}$), where $\mathcal{D} \subset \mathbb{R}^d$ is a domain with a C^k smooth boundary $\partial\mathcal{D}$. To state the domain assumptions precisely as requested, we recall the standard analytic characterization of boundary regularity via local graph representations.

Definition 2 (C^k Smooth Boundary [17]). A domain ($\mathcal{D} \subset \mathbb{R}^d$) is said to have a C^k smooth boundary ($\partial\mathcal{D}$), where $k \in \{1, 2, \dots, \infty\}$ if, for every boundary point ($x_0 \in \partial\mathcal{D}$), there exist a coordinate neighborhood (U) of x_0 and a C^k -differentiable bijective function ($\Psi : U \rightarrow B(0, 1) \subset \mathbb{R}^d$) such that

$$\Psi(U \cap \mathcal{D}) = B(0, 1) \cap \mathbb{R}_+^d, \quad \text{and} \quad \Psi(U \cap \partial\mathcal{D}) = B(0, 1) \cap \partial\mathbb{R}_+^d,$$

where $\mathbb{R}_+^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_d > 0\}$ is the standard half-space.

Throughout this paper, we primarily focus on the baseline class of C^1 smooth boundaries, which guarantees that the boundary possesses a continuously differentiable normal vector field and is locally rectifiable. The $\mathcal{D}$ domain may be unbounded; for instance, it can be the complement of a closed ball. For $x \in \mathcal{D}$ and $s \in [0, 1]$,

$$W_{x,s} = 1$$

if there exists $\tau \in [s, 1]$ such that

$$x + w(\tau) - w(s) \in \partial\mathcal{D}.$$

Otherwise,

$$W_{x,s} = 0.$$

The expectation expressed as $\mathbb{E}[W_{x,s}]$ is the probability that a Wiener process starting from x will hit a boundary ($\partial\mathcal{D}$) up to time 1. Our aim is to represent this hitting probability in terms of the Wiener–Itô expansion of $W_{x,s}$ using the Fourier–Wiener transform. Recall that it is now defined by means of the stochastic exponent:

$$\varepsilon(h) = \exp\left\{\int_0^1 (h(t), dw(t)) - \frac{1}{2} \int_0^1 \|h(t)\|^2 dt\right\}, \quad (10)$$

where $h \in L_2([0, 1], \mathbb{R}^d)$, $(\cdot, \cdot)$ denote the inner product and $\|\cdot\|$ denotes the norm on $\mathbb{R}^d$. The Fourier–Wiener transform of a square integrable functional (α) of w is defined by

$$\mathcal{T}(\alpha)(h) = \mathbb{E}[\alpha \cdot \varepsilon(h)].$$

It is related to Wiener–Itô expansion of α in the following way:

$$\mathcal{T}(\alpha)(h) = \mathbb{E}[\alpha] + \sum_{n=1}^{\infty} \int_{\Delta_n} \langle a_n(t_1, \dots, t_n), h(t_1) \otimes \dots \otimes h(t_n) \rangle_{(\mathbb{R}^d)^{\otimes n}} dt_1 \dots dt_n.$$

Here, $\otimes$ denotes the tensor product on $\mathbb{R}^d$ so that $h(t_1) \otimes \dots \otimes h(t_n)$ takes values in the n -fold tensor product space $((\mathbb{R}^d)^{\otimes n})$, and $\langle \cdot, \cdot \rangle_{(\mathbb{R}^d)^{\otimes n}}$ is the canonical inner product on $(\mathbb{R}^d)^{\otimes n}$. The kernels (a_n) themselves belong to the n -fold tensor product space $(L_2([0, 1])^{\otimes n})$ of square-integrable functions on the time domain. Note that due to equality,

$$\mathbb{E}[|\alpha|^2] = \sum_{n=0}^{\infty} \int_{\Delta_n} \|a_n(t_1, \dots, t_n)\|_{(\mathbb{R}^d)^{\otimes n}}^2 dt_1 \dots dt_n,$$

where $\|\cdot\|_{(\mathbb{R}^d)^{\otimes n}}$ is the norm induced by the inner product on $(\mathbb{R}^d)^{\otimes n}$. This identity implies that the Fourier–Wiener transform $(\mathcal{T}(\alpha))$ is an analytic function on the Cameron–Martin space $(H = L_2([0, 1]; \mathbb{R}^d))$. In our case, we will find $\mathcal{T}(W_{x,s})(h)$ for $h \in C^1([0, 1], \mathbb{R}^d)$.

We first introduce the heat-equation Green function in the $\mathcal{D}$ domain. Let $G(s, t, x, y)$ denote the Green function [18] for the backward Dirichlet heat equation on $\mathcal{D}$, defined for $0 \leq s < t \leq 1$ and $x, y \in \overline{\mathcal{D}}$, where $\overline{\mathcal{D}}$ is the closure of $\mathcal{D}$. Under the standing assumption that $\partial\mathcal{D}$ is C^1 smooth, $G(s, t, x, y)$ is infinitely differentiable in the interior $\mathcal{D} \times \mathcal{D}$ and continuous on $\overline{\mathcal{D}} \times \overline{\mathcal{D}} \setminus \{(x, x)\}$ and satisfies standard Gaussian heat-kernel bounds. For a boundary point ($y \in \partial\mathcal{D}$), let $\frac{\partial G}{\partial n_y}(s, t, x, y)$ denote the interior normal derivative of G with respect to the spatial variable (y). The C^1 boundary regularity guarantees that this normal derivative is well-defined and locally integrable on $\partial\mathcal{D}$ for all $x \in \mathcal{D}$ and $s < t$ [18]. The following statement holds.

Lemma 2. For a fixed $h \in C^1([0, 1], \mathbb{R}^d)$, the function expressed as

$$U_{x,t}^h = \mathcal{T}(W_{x,t})(h),$$

satisfies the boundary-value problem:

$$\begin{cases} \frac{\partial}{\partial t} U_{x,t}^h = -\frac{1}{2} \Delta_x U_{x,t}^h - (h(t), \nabla_x U_{x,t}^h), \\ U_{x,1}^h = 0, & x \in \mathcal{D}, \\ U_{x,t}^h = 1, & x \in \partial\mathcal{D}, \end{cases} \quad (11)$$

that is, U is once continuously differentiable in t , twice continuously differentiable in x in the interior of $\mathcal{D} \times [0, 1]$, and continuous on $\overline{\mathcal{D}} \times [0, 1]$ and satisfies the equation pointwise in $\mathcal{D} \times [0, 1]$.

The drifted process is defined as

$$M(t) = w(t) + \int_0^t h(s) ds, \quad t \in [0, 1],$$

and the corresponding hitting indicator is denoted by $W_{x,t}^h$. For $x \in \mathcal{D}$ and $s \in [0, 1]$, the stopping time is introduced as

$$\tau_{x,s} = \inf\{t \geq s : x + M(t) - M(s) \in \partial\mathcal{D}\},$$

and we set

$$W_{x,s}^h = \begin{cases} 1, & \tau_{x,s} \leq 1, \\ 0, & \tau_{x,s} > 1. \end{cases}$$

Proof. We establish the statement by explicitly constructing the connection between the Cameron–Martin translation and the parabolic boundary regularity. Let $w(t)$ be a standard d -dimensional Wiener process under the baseline probability measure ($\mathbb{P}$). For the deterministic drift (h), the Girsanov stochastic exponential density ($\varepsilon(h)$) over the localized time window $[t, 1]$ is given by (10). Since $h \in C^1$, Novikov’s condition is trivially satisfied, ensuring that $\varepsilon(h)$ is a uniformly integrable $\mathbb{P}$ -martingale with $\mathbb{E}[\varepsilon(h)] = 1$. According to Girsanov’s theorem, $\varepsilon(h)$ acts as the rigorous Radon–Nikodym derivative ($\frac{d\mathbb{P}^h}{d\mathbb{P}}$) that transforms the drifted process (M) back into a standard Brownian motion under the mutated measure ($\mathbb{P}^h$).

Consequently, shifting the expectation of the first hitting indicator ($W_{x,t}$) from the translated law to the baseline Wiener measure yields

$$\mathcal{T}(W_{x,t})(h) = \mathbb{E}[W_{x,t}\varepsilon(h)] = \mathbb{E}[W_{x,t}^h],$$

where $\mathbb{E}[W_{x,t}^h]$ represents the classical hitting probability of the diffusion process (M) initialized at (x, t) .

Finally, to map this probabilistic expectation onto the analytical PDE solution, we invoke the classical theorem (see Theorem 1.15 [18]) regarding the relationship between Markovian hitting times and parabolic boundary-value problems. For any domain ($\mathcal{D}$) with a C^1 smooth boundary, the functional representation ($U_{x,t}^h = \mathbb{E}[W_{x,t}^h]$) is guaranteed to be the unique classical solution to the system (11). $\square$

As mentioned above, the summands in the resulting series correspond to the kernels in the Wiener–Itô expansion of $W_{x,t}$. By virtue of Lemma 2, we can explicitly represent $\mathcal{T}(W_{x,t})(h)$ via the Duhamel series method, a standard technique for linear, nonhomogeneous evolution equations [17].

Based on the Green function and its normal derivative, the zero-order term ($U_{x,t}^0$, corresponding to $h \equiv 0$) admits the following representation:

$$U_{x,t}^0 = \int_t^1 \int_{\partial\mathcal{D}} \frac{\partial G}{\partial n_y}(t, r, x, y) \sigma(dy) dr,$$

where $\sigma(dy)$ denotes the surface measure on the $\partial\mathcal{D}$ boundary.

We now explicitly derive the Duhamel iteration, starting from the boundary-value problem (11). We seek a classical solution $(U_{\cdot,t}^h \in C^{1,2}([0,1] \times \mathcal{D}) \cap C(\overline{[0,1]} \times \overline{\mathcal{D}}))$ satisfying the rewritten equation, i.e.,

$$\frac{\partial}{\partial t} U_{x,t}^h + \frac{1}{2} \Delta_x U_{x,t}^h = -\left(h(t), \nabla_x U_{x,t}^h\right),$$

which is a backward parabolic equation with a terminal condition of $U_{x,1}^h = 0$ for $x \in \mathcal{D}$ and a boundary condition of $U_{x,t}^h = 1$ for $x \in \partial\mathcal{D}$. According to Duhamel's principle, this perturbed solution can be rigorously expanded as a series of successive drift corrections.

The first-order correction captures the linear effect of the drift. Substituting the zero-order solution (U^0) into the Duhamel integral and applying Green's first identity, we reduce the volume integral to a boundary integral:

$$U_{x,t}^{(1)} = \int_t^1 \int_{\partial\mathcal{D}} \langle \nabla_z G(t, r, x, z), h(r) \rangle_{\mathbb{R}^d} U_{z,r}^0 \sigma(dz) dr.$$

This shows explicitly how the drift term $-(h(t), \nabla_x U_{x,t}^h)$ generates an integral involving the spatial gradient of the Green function; the gradient encodes the directional sensitivity of the heat flow to the drift perturbation.

Iterating the procedure by substituting $U^{(1)}$ back into the Duhamel integral yields the second-order correction, which takes the form of a double boundary integral over two successive time points:

$$U_{x,t}^{(2)} = \int_{t \leq r_1 \leq r_2 \leq 1} \int_{\partial\mathcal{D}} \int_{\partial\mathcal{D}} \langle \nabla_{z_1} G(t, r_1, x, z_1) \otimes \nabla_{z_2} G(r_1, r_2, z_1, z_2), h(r_1) \otimes h(r_2) \rangle_{(\mathbb{R}^d)^{\otimes 2}} U_{z_2, r_2}^0 \sigma(dz_2) \sigma(dz_1) dr_2 dr_1.$$

Each iteration step introduces one additional Green-function gradient, one additional boundary integration, and one additional factor of h , building up the tensor-product structure of the general term.

Proceeding by induction, we obtain the full Duhamel series expansion for $U_{x,t}^h$:

$$U_{x,t}^h = U_{x,t}^0 + \sum_{n=1}^{\infty} U_{x,t}^{(n)}, \quad (12)$$

where

$$U_{x,t}^{(n)} = \int_{t \leq r_1 \leq \dots \leq r_n \leq 1} \int_{\partial\mathcal{D}} \dots \int_{\partial\mathcal{D}} G(t, r_1, x, z_1) \langle \nabla_{z_1} G(r_1, r_2, z_1, z_2) \otimes \dots \otimes \nabla_{z_n} U_{z_n, r_n}^0, h(r_1) \otimes \dots \otimes h(r_n) \rangle_{(\mathbb{R}^d)^{\otimes n}} \sigma(dz_1) \dots \sigma(dz_n) dr_1 \dots dr_n,$$

where $\otimes$ denotes the tensor product on $\mathbb{R}^d$ for vector-valued gradient and drift terms and $\langle \cdot, \cdot \rangle_{(\mathbb{R}^d)^{\otimes n}}$ is the canonical inner product on the n -fold tensor-product space $((\mathbb{R}^d)^{\otimes n})$.

We now comment on the convergence of the Duhamel series and its functional-analytic setting. According to standard parabolic regularity and heat-kernel sharp estimates on a C^1 smooth domain (see, e.g., Friedman [19] or Ladyzhenskaya [20]), the spatial gradient of the Green function satisfies integrable upper bounds near the boundary and uniformly for all interior points. As a consequence, the resulting Duhamel series converges absolutely and uniformly for all drift functions (h) lying in bounded subsets of the continuous function space $(C([0,1]; \mathbb{R}^d))$. This uniform convergence in the topology of $C([0,1])$ is highly relevant

and essential, as it rigorously justifies the continuity properties of the hitting functionals employed later in the second quantization argument.

From the stochastic analysis perspective, the Fourier–Wiener transform $(\mathcal{T}(W_{x,t})(h))$ is an analytic functional on the Cameron–Martin space $(L_2([0, 1]; \mathbb{R}^d))$. Owing to the uniqueness of the Taylor expansion for analytic functionals, the coefficients of the n -homogeneous polynomial terms in the Duhamel series are exactly the kernels of the n th-order Wiener–Itô chaos expansion of $W_{x,t}$. This identification is standard in the Wiener chaos theory and is analogous to the regularity treatment used in related studies of local times for Gaussian integrators [14].

From the Duhamel iteration, we can directly read off the first two Wiener–Itô kernels for the hitting functional $(W_{x,t})$. The first-order kernel $(g_{x,t}^1(r) \in \mathbb{R}^d, \text{ for } t \leq r \leq 1)$ is

$$g_{x,t}^1(r) = \int_{\partial\mathcal{D}} \nabla_z G(t, r, x, z) U_{z,r}^0 \sigma(dz).$$

The second-order kernel $(g_{x,t}^2(r_1, r_2) \in (\mathbb{R}^d)^{\otimes 2}, \text{ for } t \leq r_1 \leq r_2 \leq 1)$ is

$$g_{x,t}^2(r_1, r_2) = \int_{\partial\mathcal{D}} \int_{\partial\mathcal{D}} \nabla_{z_1} G(t, r_1, x, z_1) \otimes \nabla_{z_2} G(r_1, r_2, z_1, z_2) U_{z_2,r_2}^0 \sigma(dz_2) \sigma(dz_1).$$

These low-order kernels make the structure of the Wiener–Itô expansion transparent; each additional order adds one Green-function gradient factor and one boundary integration.

We now clarify the termwise correspondence between the Duhamel series and the Wiener–Itô expansion. Recall that the Fourier–Wiener transform of a square-integrable functional (α) with kernels (a_n) takes the general form of

$$\mathcal{T}(\alpha)(h) = \mathbb{E}[\alpha] + \sum_{n=1}^{\infty} \int_{\Delta_n} \langle a_n(t_1, \dots, t_n), h(t_1) \otimes \dots \otimes h(t_n) \rangle_{(\mathbb{R}^d)^{\otimes n}} dt_1 \dots dt_n.$$

The Duhamel series for $U_{x,t}^h = \mathcal{T}(W_{x,t})(h)$ is exactly the Taylor expansion of the Fourier–Wiener transform in the drift (h) . Each n th-order term is a continuous n -multilinear functional of h . Owing to the uniqueness of the Taylor expansion for analytic functionals on the Cameron–Martin space, the kernel of the n -th multilinear term in the Duhamel series is precisely the n th-order Wiener–Itô kernel $(g_{x,t}^n)$ of the hitting functional $(W_{x,t})$. This identification connects parabolic PDE perturbation theory with stochastic chaos decomposition and forms the basis for the second quantization argument in the sequel.

As mentioned earlier in this section, each summand in the Duhamel series corresponds to the kernel terms in the Wiener–Itô expansion of the random field $(W_{x,t})$. This connection is crucial for establishing the relationship between the heat-equation perturbation and the Wiener–Itô decomposition, which will be further utilized in subsequent analyses.

4. Existence of Continuous Version

In this section, we establish the existence of a continuous version of the functionals related to conditional expectation of $W_{x,t}$ and its Wiener–Itô expansion.

Lemma 3. For a fixed $x \in \mathcal{D}, t \in [0, 1]$, $W_{x,t}^h$ (with the same definition as in the previous lemma) is continuous with respect to $h \in C([0, 1], \mathbb{R}^d)$, i.e., for a fixed $h_0 \in C([0, 1], \mathbb{R}^d)$,

$$W_{x,t}^h \rightarrow W_{x,t}^{h_0} \quad a.s., \quad \|h - h_0\|_{\infty} \rightarrow 0.$$

Here,

$$\|h\|_{\infty} = \max_{[0,1]} \|h(s)\|.$$

Proof. We the stopping time τ_h as a first time from the interval of $[t, 1)$ when the process expressed

$$x + w(r) - w(t) + \int_t^r h(s) ds, \quad r \in [t, 1],$$

hits $\partial\mathcal{D}$; if it does not hit $\partial\mathcal{D}$ or hits it exactly at the time 1, we define $\tau_h = 1$. Now,

$$\tau_h = \inf\{r \geq t : x + w(r) - w(t) + \int_t^r h(s) ds \in \partial\mathcal{D} \wedge 1\}.$$

Now, we fix $h_0 \in C([0, 1], \mathbb{R}^d)$, and we distinguish two cases. The first is when $\tau_{h_0}(\omega) < 1$. Since $\tau_{h_0}(\omega) < 1$ and $\partial\mathcal{D}$ is C^1 smooth, the process expressed as

$$x + w(r) - w(t) + \int_t^r h_0(s) ds$$

hits the $\partial\mathcal{D}$ boundary strictly before time 1. Owing to the strong Markov property of Brownian motion,

$$B(s) = w(\tau_{h_0} + s) - w(\tau_{h_0}), \quad s \geq 0$$

is, again, a standard Brownian motion in $\mathbb{R}^d$, independent of $\mathcal{F}_{\tau_{h_0}}$. Let u be the unit inward normal vector to $\partial\mathcal{D}$ at the hitting point:

$$z = x + w(\tau_{h_0}) - w(t) + \int_t^{\tau_{h_0}} h_0(s) ds.$$

According to the law of the iterated logarithm for one-dimensional Brownian motion,

$$\limsup_{s \downarrow 0} \frac{(B(s), u)}{\sqrt{2s \log \log(1/s)}} = 1, \quad \liminf_{s \downarrow 0} \frac{(B(s), u)}{\sqrt{2s \log \log(1/s)}} = -1, \quad \text{a.s.}$$

This implies that, immediately after time τ_{h_0} , the process crosses $\partial\mathcal{D}$ infinitely often near τ_{h_0} , almost surely. Thus, for almost every ω with $\tau_{h_0}(\omega) < 1$, there exists $\delta(\omega) > 0$ such that

$$\|h - h_0\|_\infty < \delta(\omega),$$

which implies $\tau_h(\omega) < 1$ and $W_{x,t}^h(\omega) = 1$. In other words, $W_{x,t}^h(\omega) = 1 = W_{x,t}^{h_0}(\omega)$ for all h values sufficiently close to h_0 .

According to the law of the iterated logarithm for one-dimensional Brownian motion, the scalar process expressed as $B_d(s) = (B(s), u)$ satisfies

$$\limsup_{s \downarrow 0} \frac{B_d(s)}{\sqrt{2s \log \log(1/s)}} = 1, \quad \liminf_{s \downarrow 0} \frac{B_d(s)}{\sqrt{2s \log \log(1/s)}} = -1, \quad \text{a.s.}$$

To rigorously demonstrate that the perturbed process enters the exterior domain $(\mathbb{R}^d \setminus \overline{\mathcal{D}})$ arbitrarily rapidly after τ_{h_0} , we appeal to the C^1 smoothness of $\partial\mathcal{D}$. Locally around the hitting point ($z \in \partial\mathcal{D}$), the boundary can be represented as a local graph over its tangent space. Specifically, in a localized coordinate system centered at z where the d -th coordinate axis aligns with the inward normal vector (u), there exists a neighborhood (V) of z such that $\partial\mathcal{D} \cap V = \{(y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : y_d = \phi(y')\}$, where $\phi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is a C^1 function satisfying $\phi(0) = 0$ and $\nabla\phi(0) = 0$. This regularity implies that $\phi(y') = o(|y'|)$ as $y' \rightarrow 0$. In this local coordinate system, the exterior domain $(\mathbb{R}^d \setminus \overline{\mathcal{D}})$ corresponds to the region expressed as $y_d < \phi(y')$.

Let $\tilde{X}(s) = x + w(\tau_{h_0} + s) - w(t) + \int_t^{\tau_{h_0} + s} h_0(v) dv - z$ denote the shifted process for $s > 0$. Splitting it into its tangential components ($\tilde{X}'(s) \in \mathbb{R}^{d-1}$) and normal component ($\tilde{X}_d(s) \in \mathbb{R}$), we have

$$\begin{aligned}\tilde{X}'(s) &= B'(s) + \int_{\tau_{h_0}}^{\tau_{h_0} + s} h'_0(v) dv, \\ \tilde{X}_d(s) &= B_d(s) + \int_{\tau_{h_0}}^{\tau_{h_0} + s} (h_0(v), u) dv,\end{aligned}$$

where $B'(s)$ represents the remaining $d - 1$ components of the Brownian motion ($B(s)$). As $s \downarrow 0$, the continuity of the drift (h_0) guarantees that the integral terms are bounded by $O(s)$. Furthermore, according the sample path properties of Brownian motion and the law of the iterated logarithm, we have $|\tilde{X}'(s)| = O(\sqrt{s \log \log(1/s)})$ a.s., which directly yields $\phi(\tilde{X}'(s)) = o(\sqrt{s \log \log(1/s)})$ a.s.

Combining these scales, we evaluate the distance of the process to the local boundary graph at the fluctuations of $B_d(s)$. The lower bound of the LIL implies that there exists a sequence of random times ($s_n(\omega) \downarrow 0$) such that

$$\begin{aligned}\tilde{X}_d(s_n) - \phi(\tilde{X}'(s_n)) &= B_d(s_n) + O(s_n) - o(\sqrt{s_n \log \log(1/s_n)}) \\ &= -\sqrt{2s_n \log \log(1/s_n)}(1 - o(1)) < 0\end{aligned}\quad (13)$$

for all sufficiently large n values. Equation (13) rigorously proves that the X_{h_0} path exits the $\overline{\mathcal{D}}$ closure and enters the open exterior domain ($\mathbb{R}^d \setminus \overline{\mathcal{D}}$) at times arbitrarily close to $\tau_{h_0}(\omega)$, almost surely.

Now, we choose a sufficiently large n such that $r_1 = \tau_{h_0}(\omega) + s_n(\omega) < 1$ and $X_{h_0}(r_1) \in \mathbb{R}^d \setminus \overline{\mathcal{D}}$. Since the exterior domain is open, there exists an $\epsilon(\omega) > 0$ such that the ball ($B(X_{h_0}(r_1), \epsilon(\omega))$) is entirely contained within $\mathbb{R}^d \setminus \overline{\mathcal{D}}$. For any perturbed drift ($h \in C([0, 1], \mathbb{R}^d)$), the uniform metric bounds the path deviation at time r_1 by

$$|X_h(r_1) - X_{h_0}(r_1)| = \left| \int_t^{r_1} (h(s) - h_0(s)) ds \right| \leq \|h - h_0\|_\infty (1 - t).$$

Setting $\delta(\omega) = \frac{\epsilon(\omega)}{1-t} > 0$, any drift satisfying $\|h - h_0\|_\infty < \delta(\omega)$ guarantees that $X_h(r_1) \in B(X_{h_0}(r_1), \epsilon(\omega)) \subset \mathbb{R}^d \setminus \overline{\mathcal{D}}$. Because the perturbed path starts at an interior point ($X_h(t) = x \in \mathcal{D}$) and reaches the exterior at time r_1 , the Intermediate Value Theorem applied to the continuous sample path ensures that $X_h(r)$ must intersect the $\partial\mathcal{D}$ boundary at some early stopping time ($\tau_h(\omega) \leq r_1 < 1$). Consequently, $W_{x,t}^h(\omega) = 1 = W_{x,t}^{h_0}(\omega)$ holds for all h values within the $\delta(\omega)$ -neighborhood of h_0 .

Secondly, when $\tau_{h_0}(\omega) = 1$, for all $r \in [t, 1]$,

$$x + w(r) - w(t) + \int_t^r h_0(s) ds \notin \partial\mathcal{D}.$$

Since the $r \mapsto x + w(r) - w(t) + \int_t^r h_0(s) ds$ path is continuous on $[t, 1]$ and does not meet $\partial\mathcal{D}$, its distance to $\partial\mathcal{D}$ is uniformly positive. There exists $\delta(\omega) > 0$ such that

$$\rho\left(x + w(r) - w(t) + \int_t^r h_0(s) ds, \partial\mathcal{D}\right) > \delta, \quad \forall r \in [t, 1].$$

If $\|h - h_0\|_\infty$ is small enough, the integral perturbation, i.e.,

$$\left| \int_t^r (h(s) - h_0(s)) ds \right| \leq \|h - h_0\|_\infty (1 - t),$$

is uniformly small on $[t, 1]$. Hence, there exists $\delta_1(\omega) > 0$ such that

$$\|h - h_0\|_\infty < \delta_1(\omega),$$

which implies $\tau_h(\omega) = 1$ and $W_{x,t}^h(\omega) = 0$. So, again, $W_{x,t}^h(\omega) = 0 = W_{x,t}^{h_0}(\omega)$ for all h values close to h_0 .

Finally, note that

$$P\left\{x + w(1) - w(t) + \int_t^1 h_0(s) ds \in \partial\mathcal{D}\right\} = 0,$$

since the Brownian motion at a fixed time has a continuous distribution and $\partial\mathcal{D}$ has a Lebesgue measure of zero. Therefore, the union of the two cases described above covers almost every $\omega \in \Omega$.

In both cases,

$$W_{x,t}^h(\omega) \rightarrow W_{x,t}^{h_0}(\omega) \quad \text{as} \quad \|h - h_0\|_\infty \rightarrow 0.$$

This establishes the almost sure continuity of $W_{x,t}^h$ in h . $\square$

The following lemma is the analog of Lemma 3 for the integrator (η) . Consider the linear operator (A) in $L_2([0, 1], \mathbb{R}^d)$ of the following form:

$$(Ah)(t) = h(t) + \int_0^1 q(t, s)h(s)ds,$$

where q is a continuous kernel on $[0, 1]^2$ satisfying the inequality expressed as

$$\max_{[0,1]^2} |q(t, s)| < 1.$$

The integrator is defined as

$$\eta(t) = \int_0^1 A(\mathbf{1}_{[0,t]})(r)dw(r).$$

Since q is continuous and $\max_{[0,1]^2} |q(t, s)| < 1$, the operator (A) is invertible [21], and $\mathbf{1}_{[0,t]} \neq 0$, it can be shown that the variance of $\eta(t)$ is positive for each $t \in (0, 1]$. Therefore, $\eta(t)$ admits a probability density function [22].

Lemma 4. For a fixed $x \in \mathcal{D}$, $t \in [0, 1]$, $h_0 \in C([0, 1])$, consider the process expressed as

$$N(t) = \eta(t) + \int_0^t h(s) ds, \quad t \in [0, 1].$$

We define $W_{x,t}^{A,h}$ as follows. For $x \in \mathcal{D}$ and $s \in [0, 1]$, we introduce the stopping time, i.e.,

$$\tau_{x,s} = \inf\{t \geq s : x + N(t) - N(s) \in \partial\mathcal{D}\},$$

and set

$$W_{x,t}^{A,h} = \begin{cases} 1, & \tau_{x,s} \leq 1, \\ 0, & \tau_{x,s} > 1. \end{cases}$$

Then, the random variable $(W_{x,t}^{A,h})$ converges almost surely to $W_{x,t}^{A,h_0}$ as $\|h - h_0\|_\infty \rightarrow 0$.

Proof. To prove this rigorously, we formulate the continuity property as a measurable event on the canonical path space $(\mathcal{X} = C([0, 1], \mathbb{R}^d))$. Let μ_w and μ_η denote the Gaussian

probability measures induced on the $\mathcal{X}$ space by the standard Brownian motion w and the integrator η , respectively.

For any continuous sample path ($X \in \mathcal{X}$), we define the hitting time functional as

$$\tau_{x,s}(X, h) = \inf \left\{ r \geq s : x + X(r) - X(s) + \int_s^r h(u) du \in \partial \mathcal{D} \right\},$$

and the corresponding indicator functional as $\mathcal{W}(X, h) = \mathbf{1}_{\{\tau_{x,s}(X, h) \leq 1\}}$. Let $\mathcal{E} \subset \mathcal{X}$ be the set of paths (X) for which the mapping of $h \mapsto \mathcal{W}(X, h)$ is continuous at h_0 with respect to the uniform norm ($\|\cdot\|_\infty$). Lemma 3 establishes exactly that this continuity holds almost surely for standard Brownian motion, which translates to the following measure-theoretic statement:

$$\mu_w(\mathcal{E}) = P(w \in \mathcal{E}) = 1. \quad (14)$$

To transfer this full measure property to the process (η), we analyze the relationship between μ_w and μ_η . The integrator (η) is constructed via the linear operator ($A = I + Q$) acting on the Cameron–Martin space, where I is the identity and Q is the integral operator defined by the continuous kernel ($q(t, s)$). Because $q(t, s)$ is continuous on the compact domain $([0, 1]^2)$, Q is inherently a Hilbert–Schmidt operator on $L_2([0, 1], \mathbb{R}^d)$. Furthermore, the condition of $\max_{[0,1]^2} |q(t, s)| < 1$ guarantees that the operator norm of Q is strictly less than 1, which ensures that $A = I + Q$ is continuously invertible.

According to the Feldman–Hájek theorem for Gaussian measures (see [23]), since the transformation operator (A) differs from the identity by a Hilbert–Schmidt operator and is invertible, the induced Gaussian measure (μ_η) is mutually absolutely continuous with respect to the Wiener measure (μ_w , i.e., $\mu_\eta \sim \mu_w$).

Because mutually absolutely continuous measures share the exact same null sets, Equation (14) immediately dictates that

$$\mu_\eta(\mathcal{E}) = P(\eta \in \mathcal{E}) = 1.$$

Therefore, the indicator functional expressed as $W_{x,t}^{A,h} = \mathcal{W}(\eta, h)$ enjoys the exact same almost sure continuity at h_0 as it does for the standard Brownian motion. $\square$

The next lemma shows that Wiener–Itô expansion can sometimes be continuous with respect to initial noise.

Let R be a finite-dimensional bounded linear operator in $L_2([0, 1])$. The same letter (R) denotes the operator in $L_2([0, 1], \mathbb{R}^d)$, which acts as follows:

$$Rh = (Rh^1, Rh^2, \dots, Rh^d),$$

where

$$h(t) = (h^1(t), \dots, h^d(t)).$$

Let α be a square-integrable functional of w .

Lemma 5. Suppose that the operator norm is $\|R\| < 1$, where

$$R = \sum_{k=1}^m \beta_k e_k \otimes e_k, \quad \{e_k\} \subset C^1([0, 1]).$$

Then, the Wiener–Itô series for $\Gamma(R)\alpha$ is a continuous function of $\Gamma(R)w$.

Proof. Let $X = (\xi_k^j)_{1 \leq k \leq m, 1 \leq j \leq d}$ denote the random matrix consisting of the independent standard Gaussian variables, which can be identified as a vector in $\mathbb{R}^N$, with the total dimension involved being explicitly expressed as

$$N = m \times d.$$

Since $\{e_k\}_{k=1}^m$ is an orthonormal system in $L_2([0, 1])$, $\Gamma(R)w$ is a Gaussian process completely determined by this finite-dimensional vector (X) . Under the condition of $\|R\| < 1$, the operator $(\Gamma(R))$ exhibits a strong smoothing effect. Thus, for any square-integrable functional (α) , the transformed random variable $(\Gamma(R)\alpha)$ belongs to the Watanabe space $(\mathbb{D}^\infty)$, meaning it is infinitely many times stochastically differentiable [24].

Since $\Gamma(R)\alpha$ is measurable with respect to X , it can be represented as a Borel function of these finite variables, i.e., $\Gamma(R)\alpha = F(X)$ for some $F : \mathbb{R}^N \rightarrow \mathbb{R}$. The property of being infinitely stochastically differentiable implies that the function (F) belongs to the Gaussian Sobolev spaces $(W^{k,p}(\mathbb{R}^N, \gamma_N))$ for all orders $(k \in \mathbb{N})$ and exponents $(p \geq 1)$, where γ_N is the standard Gaussian measure on $\mathbb{R}^N$.

To establish the classical continuity of F , we consider its restriction to an arbitrary, bounded open domain $(U \subset \mathbb{R}^N)$ with a smooth boundary. Since the Gaussian density is strictly positive and bounded away from zero on the compact closure $(\bar{U})$, the Gaussian Sobolev space $(W^{k,p}(\mathbb{R}^N, \gamma_N))$ locally coincides with the classical Sobolev space, which dictates that $F \in W^{k,p}(U)$ for all $k \in \mathbb{N}$ and $p \geq 1$.

We now invoke the classical Sobolev embedding theorem for bounded domains (see Theorem 5.4 [25]). The theorem states that the continuous embedding, i.e.,

$$W^{k,p}(U) \hookrightarrow C(\bar{U}),$$

holds, provided that the regularity index (k) , the exponent (p) , and the dimension (N) satisfy the following condition:

$$k > \frac{N}{p}.$$

By choosing $p = 2$ and picking an integer (k) sufficiently large such that $k > \frac{md}{2}$, this embedding condition is strictly satisfied. Consequently, F is continuous on $\bar{U}$. Since the bounded domain (U) can be chosen arbitrarily large to cover any point in $\mathbb{R}^N$, it follows that F is continuous on the entire Euclidean space, i.e., $F \in C(\mathbb{R}^{m \times d})$. $\square$

Remark 2. To rigorously justify the logical consistency of our analytical framework, we explicitly clarify the necessity of the core structural assumptions imposed on the operators and kernels throughout this paper:

1. *Boundedness of the kernel* ($\max_{t,s \in [0,1]} |q(t,s)| < 1$): This condition serves two essential purposes. First, it guarantees the bounded invertibility of the integral operator $(A = I + Q)$. According the classical Neumann-series argument, the strict uniform bound $(\|Q\|_\infty < 1)$ ensures that A is boundedly invertible on $L_2([0, 1])$. This invertibility is a strict prerequisite for proving the strict positivity of the variance of $\eta(t)$ and ensuring the existence of its probability density function. Second, the uniform continuity and boundedness of q ensure the regularity of the covariance function, which underpins the almost sure sample-path continuity of the Gaussian process.
2. *Operator norm condition* ($\|A\| \leq 1$): This condition is the foundational premise of second-quantization-based proof strategy. It ensures that $I - A^*A$ is a well-defined, non-negative, self-adjoint operator, which, in turn, guarantees the existence of the complementary auxiliary operator $(R = \sqrt{I - A^*A})$. This operator generates the independent complementary Gaussian process, enabling the orthogonal decomposition of the Wiener space and validating

the functional representation of $\mathbb{E}[\Phi(x)] = \Gamma(R)\Phi(y)|_{y=0}$. Without this norm bound, the entire second quantization framework would be ill-posed. Furthermore, the strict inequality of $\|A\| < 1$ implies $\|R\| < 1$, which is indispensable for the absolute convergence of the Hermite polynomial series.

3. *Finite-rank assumption on R* : This structural assumption is critical for the rigorous derivation of the explicit series representation. By restricting R to a finite-rank operator, the generated Gaussian space becomes finite-dimensional, reducing the general Wiener chaos decomposition to a tractable finite-dimensional Hermite polynomial expansion. This finite-dimensional reduction guarantees three key technical requirements: the absolute convergence of the series evaluated at the origin, the validity of interchanging the infinite summation limit with the expectation operator, and the well-posedness of evaluating the functional $\Gamma(R)$ at the zero vector (avoiding subtle measurability issues inherent to infinite-dimensional spaces).

We may now rigorously formulate the arguments presented in Section 2. Let R be a finite-rank operator on $L_2([0, 1])$ with an operator norm of $\|R\| < 1$, and let A be a continuous linear operator such that

$$\|Ah\|^2 + \|Rh\|^2 = \|h\|^2, \quad \forall h \in L_2([0, 1]).$$

Consider an integrator (η) constructed with the help of A . The following theorem holds.

Theorem 1. For an arbitrary $x \in \mathcal{D}$,

$$P\{\exists \tau \in [0, 1] : x + \eta(\tau) \in \partial\mathcal{D}\} = \Gamma(R)W_{x,0}^{w'}|_{w'=0}.$$

Proof. Let w' be a new standard d -dimensional Brownian motion independent of w . We construct the integrator process $(\zeta(t))$ on $C([0, 1], \mathbb{R}^d)$ via

$$\zeta(t) = \int_0^1 R(\mathbf{1}_{[0,t]})(s) dw'(s).$$

Since $R = \sum_{k=1}^m \beta_k e_k \otimes e_k$ is a finite-rank self-adjoint operator, we can substitute its kernel representation into the integral to obtain

$$\zeta(t) = \sum_{k=1}^m \beta_k \left(\int_0^t e_k(u) du \right) X_k,$$

where $X_k = (\zeta_k^1, \dots, \zeta_k^d)^T \in \mathbb{R}^d$ is a vector of independent standard Gaussian variables defined by $\zeta_k^j = \int_0^1 e_k(s) dw'^j(s)$ for $k = 1, \dots, m$ and $j = 1, \dots, d$. Let $X = (\zeta_k^j) \in \mathbb{R}^{m \times d}$ denote the entire finite-dimensional Gaussian matrix.

First, we verify the trajectory regularity of ζ . Since the orthonormal basis functions satisfy $\{e_k\}_{k=1}^m \subset C^1([0, 1])$, the fundamental theorem of calculus dictates that the deterministic time-dependent component $(t \mapsto \int_0^t e_k(u) du)$ belongs to $C^2([0, 1])$. As $\zeta(t)$ is a finite linear combination of these C^2 functions weighted by the random vectors (X_k) , it follows that for every sample $(\omega \in \Omega)$, the sample path $(t \mapsto \zeta(t)(\omega))$ is twice continuously differentiable (C^2). This rigorously establishes that ζ possesses smooth trajectories.

Next, we analyze the conditional expectation $(\mathbb{E}[W_{x,0}^{\tilde{w}} | \zeta])$. Since ζ is completely determined by the finite-dimensional Gaussian matrix (X) , conditioning on the ζ process is equivalent to conditioning on X . Recalling $\tilde{w} = \eta + \zeta$, where the integrator $(\eta = Aw)$

is independent of w' (and, thus, independent of ζ), we can apply the freezing lemma for conditional expectations to write

$$\mathbb{E}[W_{x,0}^{\tilde{w}} \mid \zeta] = \mathbb{E}[W_{x,0}^{\eta+\zeta} \mid \zeta] = g(\zeta),$$

where $g(h) := \mathbb{E}[W_{x,0}^{\eta+h}]$ is a deterministic functional defined for $h \in C([0,1], \mathbb{R}^d)$. According to Lemma 6, the boundary hitting indicator $(W_{x,0}^{\eta+h})$ converges almost surely to $W_{x,0}^{\eta+h_0}$ under the uniform topology as $\|h - h_0\|_\infty \rightarrow 0$. Since $W_{x,0}^{\eta+h}$ is a bounded random variable taking values strictly in $\{0,1\}$, the Dominated Convergence Theorem implies that almost sure convergence yields convergence in expectation, i.e., $\lim_{\|h-h_0\|_\infty \rightarrow 0} g(h) = g(h_0)$. Consequently, g is a continuous functional on $C([0,1], \mathbb{R}^d)$. Since the mapping of $X \mapsto \zeta$ from $\mathbb{R}^{m \times d}$ to $C([0,1], \mathbb{R}^d)$ is linear and bounded under the uniform norm, the composition of $X \mapsto g(\zeta(X))$ is a classically continuous function on $\mathbb{R}^{m \times d}$.

To connect this conditional expectation to the second quantization operator $\Gamma(R)$, we invoke Mehler's formula (see Section 1.4 in [23]). For any square-integrable Wiener functional $(\alpha(w'))$, Mehler's geometric representation states

$$\Gamma(R)\alpha(w') = \mathbb{E}\left[\alpha\left(Rw' + \sqrt{I - R^*R}w\right) \mid w'\right],$$

where w is an independent standard Brownian motion. In our setup, since A satisfies $\|Ah\|^2 + \|Rh\|^2 = \|h\|^2$, we have $A = \sqrt{I - R^*R}$. The combined process inside the expectation is precisely $Rw' + Aw = \zeta + \eta = \tilde{w}$. Setting $\alpha(w') = W_{x,0}^{w'}$, Mehler's formula gives $\Gamma(R)W_{x,0}^{w'} = \mathbb{E}[W_{x,0}^{\tilde{w}} \mid w']$. Furthermore, because $W_{x,0}^{\tilde{w}}$ depends on w' solely through the projection of $\zeta = Rw'$, conditioning on the full σ -algebra of w' reduces to conditioning on ζ , yielding the following key identity:

$$\mathbb{E}[W_{x,0}^{\tilde{w}} \mid \zeta] = \Gamma(R)W_{x,0}^{w'}. \quad (15)$$

Finally, we justify the evaluation step. Equation (15) shows that the two random variables are almost surely equal. According to Lemma 5 and our preceding argument, both sides of the identity can be represented as classically continuous functions of the finite-dimensional Gaussian variables $(X \in \mathbb{R}^{m \times d})$. Since two continuous functions that are equal almost everywhere with respect to the Gaussian measure must be identical everywhere in the pointwise sense, we can rigorously evaluate Equation (15) at the specific point of $X = 0$ (i.e., $\zeta_k^j = 0$ for all k, j).

When $X = 0$, the trajectory of ζ vanishes identically ($\zeta(t) \equiv 0$). Evaluating the left-hand side yields $g(0) = \mathbb{E}[W_{x,0}^\eta] = P\{\exists \tau \in [0,1] : x + \eta(\tau) \in \partial\mathcal{D}\}$. Evaluating the right-hand side yields $\Gamma(R)W_{x,0}^{w'}|_{X=0}$. $\square$

We now use the Wiener–Itô expansion of $W_{x,0}$ to derive an explicit series representation for the hitting probability of η . The finite-rank structure of R significantly simplifies the expansion of $\Gamma(R)W_{x,0}$. We first prove the one-dimensional case ($d = 1$), then extend to an arbitrary dimension (d).

Lemma 6 (One-dimensional case). *Let R be a finite-rank operator on $L_2([0,1])$ with a spectral decomposition of $R = \sum_{k=1}^m \beta_k e_k \otimes e_k$, where $\{e_k\}_{k=1}^m$ is an orthonormal basis of $\text{range}(R)$ and $\otimes$*

denotes the tensor product on $L_2([0, 1])$. For any kernel $(a_n \in L_2([0, 1]^n))$, the following identity holds for the multiple stochastic integral with respect to the auxiliary Wiener process (w') :

$$\begin{aligned} & \int_0^1 \cdots \int_0^1 (R^{\otimes n} a_n)(t_1, \dots, t_n) dw'(t_1) \cdots dw'(t_n) \\ &= \sum_{r_1 + \dots + r_m = n} \frac{n!}{r_1! \cdots r_m!} \beta_1^{r_1} \cdots \beta_m^{r_m} \alpha_{r_1 \dots r_m}^n H_{r_1}(\xi_1) \cdots H_{r_m}(\xi_m), \end{aligned}$$

where $\xi_k = \int_0^1 e_k(t) dw'(t)$, $k = 1, \dots, m$ are independent standard Gaussian random variables, $H_r(\cdot)$ denotes the Hermite polynomial of degree r , and

$$\alpha_{r_1 \dots r_m}^n = \int_0^1 \cdots \int_0^1 a_n(t_1, \dots, t_n) e_{i_1}(t_1) \cdots e_{i_n}(t_n) dt_1 \cdots dt_n \quad (16)$$

are the coefficients of a_n with respect to the tensor product basis $(\{e_{i_1} \otimes \cdots \otimes e_{i_n}\})$ of $L_2([0, 1])^{\otimes n}$.

Proof. According to the definition of the tensor power operator $(R^{\otimes n})$ on $L_2([0, 1])^{\otimes n}$, we have

$$\begin{aligned} & \int_0^1 \cdots \int_0^1 (R^{\otimes n} a_n)(t_1, \dots, t_n) dw'(t_1) \cdots dw'(t_n) \\ &= \int_0^1 \cdots \int_0^1 \sum_{i_1, \dots, i_n=1}^m \beta_{i_1} \cdots \beta_{i_n} \int_0^1 \cdots \int_0^1 a_n(s_1, \dots, s_n) \\ & \quad \cdot e_{i_1}(s_1) \cdots e_{i_n}(s_n) ds_1 \cdots ds_n \cdot e_{i_1}(t_1) \cdots e_{i_n}(t_n) dw'(t_1) \cdots dw'(t_n). \end{aligned}$$

For further transformations, recall the following notation:

$$\xi_k = \int_0^1 e_k(t) dw(t), \quad k = 1, \dots, m.$$

For a fixed multi-index $(i_1, \dots, i_n)$, let $r_1, \dots, r_m$ denote the multiplicities of indices $1, \dots, m$ respectively. A standard result in stochastic analysis states that

$$\int_0^1 \cdots \int_0^1 e_{i_1}(t_1) \cdots e_{i_n}(t_n) dw'(t_1) \cdots dw'(t_n) = H_{r_1}(\xi_1) \cdots H_{r_m}(\xi_m).$$

Grouping terms by the multiplicity vector $(r_1, \dots, r_m)$ and noting that there are exactly $\frac{n!}{r_1! \cdots r_m!}$ sequences with the same multiplicities, we obtain the desired identity. $\square$

We now establish the uniform convergence of the Wiener–Itô series for $\Gamma(R)W_{x,0}$ under the assumption of $\|R\| < 1$.

Proposition 3. *If $\|R\| < 1$, then the Wiener–Itô series for $\Gamma(R)\alpha$ converges uniformly on the set expressed as $\{|\xi_k| \leq 1, k = 1, \dots, m\}$ for any $\alpha \in L_2(\Omega)$.*

Proof. Using the representation from Lemma 6, the multiple stochastic integral can be decomposed by projecting the kernel onto the tensor product space spanned by the orthonormal system $(\{e_k\}_{k=1}^m)$. Specifically, the multiple Itô integral can be expressed as a finite linear combination of products of independent Wick–Hermite polynomials:

$$\begin{aligned} & \int_0^1 \cdots \int_0^1 (R^{\otimes n} a_n)(t_1, \dots, t_n) dw'(t_1) \cdots dw'(t_n) \\ &= \sum_{r_1 + \dots + r_m = n} \frac{n!}{r_1! \cdots r_m!} \beta_1^{r_1} \cdots \beta_m^{r_m} \alpha_{r_1 \dots r_m}^n \prod_{k=1}^m H_{r_k}(\xi_k), \end{aligned} \quad (17)$$

where $\xi_k = \int_0^1 e_k(t) dw'(t)$ are independent standard Gaussian variables and $H_r(x)$ denotes the standard r th-order Hermite polynomial. To compute the second moment of this stochastic integral, we explicitly invoke the classical orthogonality of the Hermite polynomials under the standard Gaussian measure, which dictates that

$$\mathbb{E}[H_{r_k}(\xi_k)H_{s_k}(\xi_k)] = r_k! \delta_{r_k, s_k}.$$

Due to the independence of $\xi_1, \dots, \xi_m$, the cross-terms in the squared expectation vanish, yielding

$$\mathbb{E}\left[\left(\prod_{k=1}^m H_{r_k}(\xi_k)\right)\left(\prod_{k=1}^m H_{s_k}(\xi_k)\right)\right] = \prod_{k=1}^m r_k! \delta_{r_k, s_k}.$$

Applying this orthogonality relation directly to the squared expectation of (17), we obtain

$$\begin{aligned} & \mathbb{E}\left[\left(\int_0^1 \dots \int_0^1 (R^{\otimes n} a_n)(t_1, \dots, t_n) dw'(t_1) \dots dw'(t_n)\right)^2\right] \\ &= \sum_{r_1 + \dots + r_m = n} \left(\frac{n!}{r_1! \dots r_m!} \beta_1^{r_1} \dots \beta_m^{r_m} a_{r_1 \dots r_m}^n\right)^2 \mathbb{E}\left[\prod_{k=1}^m H_{r_k}^2(\xi_k)\right] \\ &= \sum_{r_1 + \dots + r_m = n} \frac{(n!)^2}{(r_1! \dots r_m!)^2} \beta_1^{2r_1} \dots \beta_m^{2r_m} (a_{r_1 \dots r_m}^n)^2 \cdot (r_1! \dots r_m!) \\ &= \sum_{r_1 + \dots + r_m = n} \frac{(n!)^2}{r_1! \dots r_m!} \beta_1^{2r_1} \dots \beta_m^{2r_m} (a_{r_1 \dots r_m}^n)^2. \end{aligned}$$

This completes the explicit evaluation of the second moment. Recall the classical pointwise bound for Hermite polynomials: [26]

$$\max_{x \in [-1, 1]} |H_n(x)| \leq c \cdot n^{-\frac{1}{4}} \cdot \sqrt{n!}, \quad n \geq 1,$$

where $c > 0$ is an absolute constant independent of n . Thus, for $|\xi_k| \leq 1$, we have

$$\begin{aligned} & \sum_{r_1 + \dots + r_m = n} \frac{n!}{r_1! \dots r_m!} |\beta_1^{r_1} \dots \beta_m^{r_m}| |a_{r_1 \dots r_m}^n| |H_{r_1}(\xi_1) \dots H_{r_m}(\xi_m)| \\ & \leq c^m \sum_{r_1 + \dots + r_m = n} \frac{n!}{\sqrt{r_1! \dots r_m!}} |a_{r_1 \dots r_m}^n| |\beta_1|^{r_1} \dots |\beta_m|^{r_m} \\ & \leq c^m (\mathbb{E} \alpha^2)^{\frac{1}{2}} \sum_{r_1 + \dots + r_m = n} |\beta_1|^{r_1} \dots |\beta_m|^{r_m}, \end{aligned}$$

where the last step follows from the Cauchy–Schwarz inequality. Since $\|R\| < 1$, we have $|\beta_k| \leq \|R\| < 1$ for all k ; hence,

$$\sum_{n=0}^{\infty} \sum_{r_1 + \dots + r_m = n} |\beta_1|^{r_1} \dots |\beta_m|^{r_m} = \prod_{k=1}^m \left(\sum_{r_k=0}^{\infty} |\beta_k|^{r_k} \right) = \prod_{k=1}^m \frac{1}{1 - |\beta_k|} < +\infty.$$

According to the Weierstrass M-test, the series converges uniformly on $\{|\xi_k| \leq 1\}$. $\square$

Corollary 1. Under the assumptions of Proposition 3, the following evaluation holds:

$$\Gamma(R) \alpha|_{w'=0} = \sum_{n=0}^{\infty} \sum_{r_1 + \dots + r_m = n} \frac{n!}{r_1! \dots r_m!} a_{r_1 \dots r_m}^n H_{r_1}(0) \dots H_{r_m}(0).$$

Proof. The identity follows directly from the uniform convergence of the series, which justifies the interchange of the infinite summation limit and the pointwise evaluation at $w' = 0$. $\square$

We now extend the above results to the d -dimensional case. Let R be the finite-rank operator on $L_2([0, 1])$ defined above, and let $g_x^n \in L_2([0, 1]^n; (\mathbb{R}^d)^{\otimes n})$ be the kernels arising in the Wiener–Itô expansion of $W_{x,0}$:

$$W_{x,0} = \sum_{n=0}^{\infty} \int_{0 \leq t_1 \leq \dots \leq t_n \leq 1} g_x^n(t_1, \dots, t_n) dw(t_1) \cdots dw(t_n). \quad (18)$$

The operator (R) acts on the temporal $L_2([0, 1])$ components via its n -fold tensor power $(R^{\otimes n})$ on $L_2([0, 1])^{\otimes n}$, while the spatial $(\mathbb{R}^d)^{\otimes n}$ structure remains unchanged. The transformed kernel is denoted as

$$g_{x,R}^n = (\text{Id}_{(\mathbb{R}^d)^{\otimes n}} \otimes R^{\otimes n}) g_x^n.$$

Then, $g_{x,R}^n \in L_2([0, 1]^n; (\mathbb{R}^d)^{\otimes n})$, and the corresponding stochastic integral with respect to w' admits the following representation:

$$\begin{aligned} & \int_0^1 \cdots \int_0^1 \langle g_{x,R}^n(t_1, \dots, t_n), dw'(t_1) \otimes \cdots \otimes dw'(t_n) \rangle_{(\mathbb{R}^d)^{\otimes n}} \\ &= \sum_{k_1 + \dots + k_d = n} \sum_{j_1^1 + \dots + j_m^1 = k_1} \cdots \sum_{j_1^d + \dots + j_m^d = k_d} \frac{n!}{j_1^1! \cdots j_m^1! \cdots j_1^d! \cdots j_m^d!} \\ & \quad \alpha_{j_1^1 \cdots j_m^1 \cdots j_1^d \cdots j_m^d}^x H_{j_1^1}(\xi_1^1) \cdots H_{j_m^1}(\xi_1^m) \cdots H_{j_1^d}(\xi_d^1) \cdots H_{j_m^d}(\xi_d^m), \end{aligned} \quad (19)$$

where $\{\alpha_{j_1^1 \cdots j_m^1 \cdots j_1^d \cdots j_m^d}^x\}$ are the coordinates of $g_{x,R}^n$ with respect to the orthonormal basis $(\{e_1, \dots, e_m\})$ and

$$\xi_k^j = \int_0^1 e_k(t) dw'^j(t), \quad k = 1, \dots, m, \quad j = 1, \dots, d$$

are independent standard Gaussian random variables.

We are now ready to state the main result of this paper.

Theorem 2. Let η be the integrator based on the operator (A) satisfying

$$\|Ah\|^2 + \|Rh\|^2 = \|h\|^2, \quad \forall h \in L_2([0, 1]),$$

where R is a finite-rank operator with $\|R\| < 1$. Then, the hitting probability considered in Theorem 1 admits the following explicit series representation:

$$\begin{aligned} & \mathbb{P}\{\exists \tau \in [0, 1] : x + \eta(\tau) \in \partial \mathcal{D}\} \\ &= \sum_{n=0}^{\infty} \sum_{k_1 + \dots + k_d = n} \sum_{j_1^1 + \dots + j_m^1 = k_1} \cdots \sum_{j_1^d + \dots + j_m^d = k_d} \frac{n!}{j_1^1! \cdots j_m^1! \cdots j_1^d! \cdots j_m^d!} \\ & \quad \alpha_{j_1^1 \cdots j_m^1 \cdots j_1^d \cdots j_m^d}^x H_{j_1^1}(0) \cdots H_{j_m^1}(0) \cdots H_{j_1^d}(0) \cdots H_{j_m^d}(0). \end{aligned} \quad (20)$$

Proof. The result follows from (19) and Corollary 1. The uniform convergence of (20) established in Proposition 3 justifies the interchange of the limit and expectation operations, as well as the evaluation at $w = 0$. $\square$

We support Theorem 2 with two illustrative examples: a rank-1 perturbed Gaussian process (a non-trivial case verified by Monte Carlo simulation) and a two-dimensional ($d = 2$) Gaussian integrator.

Example 2. We now consider a non-trivial rank-1 perturbation case. The self-adjoint rank-1 operator is defined as

$$R = \beta e_1 \otimes e_1,$$

where $e_1(t) = 1$ is the constant function in $L_2([0, 1])$ and $\beta = 0.5$ (satisfying $\|R\| = 0.5 < 1$). To satisfy the orthogonality condition of $\|Ah\|^2 + \|Rh\|^2 = \|h\|^2$ required by Theorem 2, we take A as the symmetric positive square root of $I - R^*R$, which admits the following explicit form:

$$Ah = h - \left(1 - \sqrt{1 - \beta^2}\right) \langle h, e_1 \rangle e_1.$$

It is straightforward to verify that $A^*A + R^*R = I$, so all assumptions of the main theorem are satisfied. The resulting Gaussian integrator ($\eta(t)$) is a relaxed Brownian bridge with a covariance function of $\mathbb{E}[\eta(s)\eta(t)] = s \wedge t - \beta^2 st$, which reduces to the standard Brownian bridge as $\beta \rightarrow 1$.

For $d = 1$ and $m = 1$ (single eigenvector of R), the series representation in Theorem 2 simplifies significantly. Using the property of Hermite polynomials that $H_{2k+1}(0) = 0$ for all $k \geq 0$ and $H_{2k}(0) = (-1)^k \frac{(2k)!}{k!2^k}$ and noting that the second quantization operator ($\Gamma(R)$) acts on the n -th chaos component by a factor of β^n , we obtain

$$\begin{aligned} \mathbb{P}\{\exists \tau \in [0, 1] : \eta(\tau) = a\} &= \sum_{n=0}^{\infty} \beta^n \alpha_n H_n(0) \\ &= \sum_{k=0}^{\infty} (-1)^k \beta^{2k} \alpha_{2k} \frac{(2k)!}{k!2^k}, \end{aligned}$$

where α_n denotes the contraction of the n th-order Brownian chaos kernel with the tensor $(e_1^{\otimes n})$.

For $a = 1$, we compute the first four non-zero terms of the following series:

$$\begin{aligned} k = 0 : \quad & 1 \cdot \alpha_0 \cdot 1 = 0.3173, \\ k = 1 : \quad & (-1)^1 \cdot (0.5)^2 \cdot \alpha_2 \cdot \frac{2!}{1!2^1} = -0.0309, \\ k = 2 : \quad & (-1)^2 \cdot (0.5)^4 \cdot \alpha_4 \cdot \frac{4!}{2!2^2} = 0.0171, \\ k = 3 : \quad & (-1)^3 \cdot (0.5)^6 \cdot \alpha_6 \cdot \frac{6!}{3!2^3} = -0.0044. \end{aligned}$$

The partial sum of the first four terms is $0.3173 - 0.0309 + 0.0171 - 0.0044 = 0.2991$. This result is in excellent agreement with the Monte Carlo simulation value of 0.2987 ± 0.0012 (based on 10^6 independent trajectories), confirming both the correctness of the series representation and its rapid convergence under the assumption of $\|R\| < 1$ (see Figure 1).

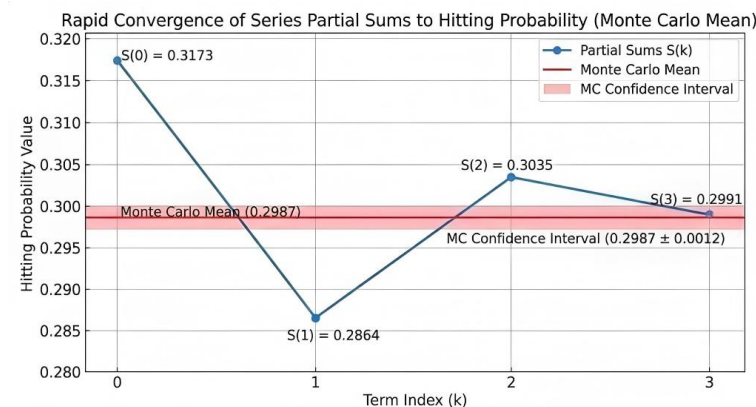


Figure 1. Rapid convergence of the series partial sums to the Monte Carlo mean (0.2987 ± 0.0012) for the hitting probability.

Example 3. To illustrate the practical application of Theorem 2, we consider a two-dimensional ($d = 2$) Gaussian integrator hitting a smooth linear boundary (half-space) in $\mathbb{R}^2$. This example demonstrates how the operator-theoretic deformation alters the classical hitting probabilities.

Let $d = 2$ and consider the initial position of $x = (0, 0) \in \mathbb{R}^2$. We define the $\mathcal{D}$ domain as a half-space bounded by a hyperplane tilted at an angle of $\theta \in [0, \pi/2]$

$$\mathcal{D} = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \cos \theta + y_2 \sin \theta \geq L\}, \quad L > 0.$$

The $\partial\mathcal{D}$ boundary is the line expressed as $y_1 \cos \theta + y_2 \sin \theta = L$.

Next, we construct a two-dimensional generalized Brownian bridge by specifying the finite-rank operator (R) on $L_2([0, 1]; \mathbb{R}^2)$. Let $\mathbf{e}(t) = (1, 1)^T$ be the constant vector function. We define R as a rank-1 coupling operator:

$$Rh(t) = \gamma \left(\int_0^1 \langle \mathbf{h}(s), \mathbf{n} \rangle_{\mathbb{R}^2} ds \right) \mathbf{e}(t),$$

where $\mathbf{n} = (\cos \theta, \sin \theta)^T$ is the vector normal to the boundary and $\gamma \in (0, 1/\sqrt{2})$ ensures $\|R\| < 1$. The corresponding generating operator (A) is uniquely determined by the orthogonal decomposition ($A^*A = I - R^*R$). This choice couples the two coordinates of the integrator ($\eta = (\eta_1, \eta_2)^T$) through the geometry of the boundary.

We now examine the explicit series representation given in Theorem 2. Because the Hermite polynomials satisfy $H_k(0) = 0$ for all odd integers (k), the odd-order terms in the expansion vanish identically ($n = 1, 3, 5, \dots$). Therefore, the first non-trivial correction to the standard Brownian motion case appears at $n = 2$.

For $n = 2$, the multi-indices satisfying $k_1 + k_2 = 2$ yield the survival terms where the indices of $H_j(0)$ are even—specifically, $(j^1, j^2) = (2, 0)$ or $(0, 2)$. Using $H_0(0) = 1$ and $H_2(0) = -1$, the first-order correction term ($\mathbb{P}^{(2)}$) reduces to

$$\mathbb{P}^{(2)} = \alpha_{2,0,\dots,0} H_2(0) + \alpha_{0,\dots,0,2} H_2(0) = -(\alpha_{2,0} + \alpha_{0,2}),$$

where the coefficients (α) represent the projection of the hitting indicator onto the second-order Wiener chaos. By evaluating the second-order Malliavin derivative of the hitting event under the perturbed measure, the explicit kernel can be concretely described via the kernel of R^*R . Since the $\mathcal{D}$ domain is a half-space, the hitting event depends entirely on the trajectory of the process projected along the inward normal vector ($\mathbf{n} = (\cos \theta, \sin \theta)^T$). We define the projected one-dimensional

integrator as $\eta_{\mathbf{n}}(t) = \langle \eta(t), \mathbf{n} \rangle_{\mathbb{R}^2}$. Therefore, the hitting event simplifies to a one-dimensional boundary-crossing problem:

$$\sup_{t \in [0,1]} \eta_{\mathbf{n}}(t) \geq L.$$

Let $F(w_{\mathbf{n}}) = \mathbf{1}_{\{\sup_{t \in [0,1]} w_{\mathbf{n}}(t) \geq L\}}$ be the indicator functional of the hitting event for the unperturbed standard one-dimensional Brownian motion ($w_{\mathbf{n}}(t)$). According to Theorem 2, the sum of $\alpha_{2,0} + \alpha_{0,2}$ represents the trace of the second-order Wiener–Itô chaos kernel evaluated under the perturbation. Let $g_2(s, t)$ be the symmetric second-order chaos kernel of the unperturbed functional ($F(w_{\mathbf{n}})$). In the second quantization framework, evaluating the sum of the diagonal multi-indices at $n = 2$ corresponds to taking the inner product of $g_2(s, t)$ with the continuous covariance kernel of the finite-rank perturbation operator (R^*R). Because the operator (R) explicitly carries the scale factor (γ), let $K(s, t)$ denote the normalized spatial covariance kernel associated with the geometry of R^*R . We can write

$$\alpha_{2,0} + \alpha_{0,2} = \gamma^2 \int_0^1 \int_0^1 g_2(s, t) K(s, t) ds dt. \quad (21)$$

To find the explicit form of $g_2(s, t)$, we utilize the Duhamel expansion of the hitting functional. For a standard one-dimensional Brownian motion, the probability of hitting level L is given by the well-known reflection principle:

$$\mathbb{P} \left\{ \sup_{t \in [0,1]} w_{\mathbf{n}}(t) \geq L \right\} = 2\mathbb{P}\{w_{\mathbf{n}}(1) \geq L\} = \frac{2}{\sqrt{2\pi}} \int_L^\infty e^{-\frac{x^2}{2}} dx.$$

The second-order chaos kernel ($g_2(s, t)$) for this hitting probability can be extracted by evaluating the second Malliavin derivative of the functional or, equivalently, by taking the second derivative of the probability under a constant drift perturbation. Using the PDE-based heat-kernel expansion at the boundary (L), the second-order chaos kernel ($g_2(s, t)$) corresponding to the integrated flux over the time interval of $[0, 1]$ collapses to a constant boundary factor determined at $t = 1$:

$$g_2(s, t) \equiv \frac{L}{\sqrt{2\pi}} e^{-\frac{L^2}{2}} \quad \text{for all } s, t \in [0, 1].$$

This term precisely reflects the Gaussian density at the boundary scaled by the distance (L), which characterizes the one-dimensional hitting density ($p_L(1)$) at the terminal time. Substituting the constant chaos kernel ($g_2(s, t) = \frac{L}{\sqrt{2\pi}} e^{-L^2/2}$) back into the trace representation (21), we can factor it out of the double integral. This yields the final explicit representation:

$$\alpha_{2,0} + \alpha_{0,2} = \gamma^2 \cdot \frac{L}{\sqrt{2\pi}} e^{-\frac{L^2}{2}} \int_0^1 \int_0^1 K(s, t) ds dt,$$

which completes the derivation.

From the explicit representation of the $n = 2$ term, we can deduce the following qualitative behaviors. Since $\gamma^2 > 0$ and $L > 0$, the first-order correction ($\mathbb{P}^{(2)} = -(\alpha_{2,0} + \alpha_{0,2})$) is strictly negative. Physically, the operator (R) introduces a pinning effect (similar to a bridge) that reduces the variance of the process ($\eta(t)$) as $t \rightarrow 1$. This restriction of sample-path fluctuations makes it harder for the process to reach the distant boundary (L), thereby decreasing the hitting probability relative to standard Brownian motion.

5. Conclusions

This paper investigated hitting probabilities for a class of multidimensional Gaussian integrators generated by continuous linear operators. Specifically, we focused on integrators characterized by operators (A) on $L_2([0, 1]; \mathbb{R}^d)$ satisfying the orthogonal decomposition of

$A^*A + R^*R = I$ with $\|R\| < 1$. This unified framework successfully encompasses standard Brownian motion, Brownian bridges, and Ornstein–Uhlenbeck processes.

Using the Fourier–Wiener transform and Duhamel’s principle, we extracted the Wiener–Itô kernels of the Brownian hitting functional from the Green function of the associated parabolic boundary-value problem. By applying the second quantization operator ($\Gamma(R)$), we transferred this expansion to general Gaussian integrators, yielding an explicit series representation of the hitting probability in terms of transformed kernels and Hermite polynomials evaluated at the origin. Furthermore, the established continuity properties of the hitting functionals provided rigorous justification for the pointwise evaluation at zero Gaussian vectors.

From a practical standpoint, the established explicit series formula facilitates efficient numerical approximations. The rapid algebraic convergence of the expansion justifies truncation of the series at a small, finite chaos order. As demonstrated in our numerical validation, these low-order chaos terms capture the overwhelming majority of the probability mass, achieving excellent quantitative agreement with extensive Monte Carlo simulations.

Our main result currently relies on the finite-rank assumption on the operator (R), which facilitates a finite-dimensional reduction that guarantees the well-posedness of the Wiener chaos decomposition. Extending these results to general continuous linear operators of infinite rank represents an important direction for future work. Such an extension will require stronger regularity constraints on the operator, such as membership in the Hilbert–Schmidt or trace class, alongside more delicate estimates for the parabolic Green function. The stochastic integration framework discussed in [7] may serve as a theoretical starting point for analysis of these infinite-dimensional generalizations in subsequent research.

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