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GAUSSIAN STRUCTURE IN COALESCING STOCHASTIC FLOWS

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In the paper, we consider the point measure that corresponds to Arratia flow. The central limit theorem of the multiple integrals with respect to this measure was obtained.

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1. Introduction.

In this paper, we consider an ordered family of standard Brownian motions $\{x(u,\cdot),\ u \in \mathbb{R}\}$, in which any two particles move independently until they meet and after that coalesce and move together. This family was constructed by R. A. Arratia [1] (it is known as the Arratia flow) as a weak limit of families of coalescing simple random walks. The formal definition is the following.

Definition 1.1. The Arratia flow is a random field $\{x(u,t),\ u \in \mathbb{R},\ t \geq 0\}$ satisfying the following conditions:

(1) for any $u \in \mathbb{R}$ the stochastic process $\{x(u,t),\ t \geq 0\}$ is a Brownian motion with respect to the common filtration

$$\mathcal{F}_t^x := \sigma\{x(u,s),\ u \in \mathbb{R},\ 0 \leq s \leq t\},\ t \geq 0,$$

starting from the point u;

(2) for any $u, v \in \mathbb{R}$

$$\text{if } u \leq v, \text{ then } x(u,t) \leq x(v,t) \text{ for all } t \geq 0;$$

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(3) for any $u, v \in \mathbb{R}$ the joint quadratic variation of $x(u,\cdot)$ and $x(v,\cdot)$ is given by

$$\langle x(u,\cdot), x(v,\cdot)\rangle_t = \int_0^t 1\!\!1_{\{x(u,s)=x(v,s)\}}\, ds, \quad t \geqslant 0,$$

where $1\!\!1_{\{\Delta\}}$ stands for the indicator function of the set Δ.

Here we should also mention that the Arratia flow can be obtained [6] as a weak limit of a flow of homeomorphisms x_ε which is defined as a solution to the stochastic differential equation with respect to $\mathbb{R}-$valued Wiener sheet on $\mathbb{R}\times[0,1]$

$$\begin{cases} dx_\varepsilon(u,t) = \int_{\mathbb{R}} \varphi_\varepsilon(x_\varepsilon(u,t)-q)W(dq,dt), \\ x_\varepsilon(u,0) = u, \end{cases}$$

where $\varphi_\varepsilon(u) = \varepsilon^{-1/2}\varphi(\varepsilon^{-1}u)^{1/2},\ u \in \mathbb{R}$ with nonnegative symmetric function $\varphi \in C_0^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \varphi(u)du = 1$.

It is known that for any time $t > 0$, the set $x(\mathbb{R},t)$ is almost surely locally finite (for various proofs, see the paper by T. Harris [15] and the monograph [7] of A. A. Dorogovtsev). From this it follows that for any time $t > 0$ and the interval $[u_1,u_2]$ the number

$$N_t([u_1,u_2]) := \#\left\{x(\mathbb{R},t)\cap[u_1,u_2]\right\}$$

of elements of the set $x(\mathbb{R},t)\cap[u_1,u_2]$ is almost surely finite. R. Tribe and O. Zaboronski [24] proved that for any $t > 0$, the random point process $x(\mathbb{R},t)$ is Pfaffian and found its kernel. Based on some of their formulae, the distribution of $\nu_t([u_1,u_2]) := \#\left\{x([u_1,u_2],t)\right\}$ was investigated by V. Fomichov in [11], for instance, the formulae for $\mathbf{P}\{\nu_t([0,u]) = k\}$ was found. For the Arratia flow, the large deviation principle and the law of the iterated logarithm for the size of the cluster containing the point zero were established by A. A. Dorogovtsev and O. V. Ostapenko [8] and A. A. Dorogovtsev, A. V. Gnedin and M. B. Vovchanskii [9] respectively.

From Definition 1.1 one can see that the joint quadratic variation of any two particles in the Arratia flow depends only on the distance between them, so $\{x(u,\cdot)-u,\ u\in\mathbb{R}\}$ is stationary with respect to the spatial variable. The formal proof of this property we give at the end of this section. In [13] the ergodicity with respect to the spatial variable was established, and an estimate for the strong mixing coefficient was found. Based on these properties, the central limit theorem of the number of clusters on large intervals was proved by E. Glinyanaya and V. Fomichov in [14]. In the present paper, the authors generalize the previous result and prove the central limit theorem for linear functionals of the point measure that represents the positions of clusters in the Arratia flow.

The structure of the paper is the following. The rest of the introduction gives the necessary preliminaries about point processes and the definition of the integral with respect to the point process which corresponds to the Arratia flow. Sec-

tion 2 contains the statement and the proof of the central limit theorem for integrals w.r.t. the point process, which corresponds to the Arratia flow at fixed moments of time $(t_1, \dots, t_k)$. In order to understand the structure of the covariance matrix in the limiting Gaussian vector, the limits of the conditional expectation $\frac{1}{\sqrt{n}}\mathbf{E}\left(\int_0^n f(u)N_t(du)\big|\mathcal{F}_s^x\right)$, $s<t$ are investigated in Section 3. Section 4 is devoted to the convergence of the integrals $\{\frac{1}{\sqrt{n}}\int_0^n f(u)N_t(du)\}_{t\in[t_0,T]}$ in a functional space. The family of limits $\{\lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_0^n f(u)N_t(du),\ f|_{[0,1]}\in L_2([0,1])\}$ defines a generalized random element in Hilbert space $L_2([0,1])$. We investigate its properties in Section 5. Here we define multi-linear forms from this Gaussian element and present the limit theorem for multiple integrals w.r.t. the Arratia point process.

Since for every $t>0$, the set $x(\mathbb{R},t)$ is countable and locally finite, we can use the following representation for the point measure

$$N_t([u_1,u_2]) := \#\left\{x(\mathbb{R},t)\cap[u_1,u_2]\right\}.$$

Denote $\{u_i\}_{i\in\mathbb{Z}} := x(\mathbb{R},t)$, then the point measure N_t can be written as

$$N_t = \sum_{i\in\mathbb{Z}}\delta_{u_i}. \tag{1.1}$$

Let us recall some basic definitions that we will use in the paper.

Definition 1.2. ([19]) $n-$th factorial power $N^{(n)}$ of a point measure $N=\sum_{i\in\mathbb{Z}}\delta_{u_i}$ is defined by

$$N^{(n)}(C) = \sum_{i_1\neq\ldots\neq i_n} 1\!\!1_{\{(u_{i_1},\ldots,u_{i_n})\in C\}},$$

where $C\in\mathcal{B}(\mathbb{R}^n)$.

By $N^{\otimes n}$ we denote the usual power of a measure N, i.e. for $\Delta_i\in\mathcal{B}(\mathbb{R})$,

$$N^{\otimes n}(\Delta_1\times\ldots\times\Delta_n) = N(\Delta_1)\ldots N(\Delta_n).$$

Definition 1.3. A locally integrable function $\rho^{(n)}:\mathbb{R}^n\to\mathbb{R}_+$ is called an $n-$point density function of a point measure N if, for any subset $A\in\mathcal{B}(\mathbb{R}^n)$ the following formula holds

$$\mathbf{E}N^{(n)}(A) = \int_A \rho^{(n)}(\vec{x})d\vec{x}.$$

For example, the Poisson process on the line with intensity measure $\mu(dx)=p(x)dx$ has $n-$point density function that is equal to (see, for example, Corollary 4.10 in [19])

$$\rho^{(n)}(\vec{x}) = p^{\otimes n}(\vec{x}) = p(x_1)p(x_2)\ldots p(x_n).$$

In the papers [21], [24] the authors proved existence of the $n-$point density functions $\rho_t^{(n)}$ for the point measure N_t. For instance,

$$\rho_t^{(1)}(v) = \frac{1}{\sqrt{\pi t}}, \tag{1.2}$$

$$\rho_t^{(2)}(v_1, v_2) = \frac{1}{\pi t}\left(1 + \frac{|v_2 - v_1|}{2\sqrt{t}} \cdot e^{-(v_2-v_1)^2/4t} \cdot \int\limits_{|v_2-v_1|/\sqrt{t}}^{+\infty} e^{-v^2/4}\, dv - e^{-(v_2-v_1)^2/2t}\right). \tag{1.3}$$

The connection between integrals with respect to the usual power of measure and factorial power is given in the next lemma. The proof of this statement is combinatorial and we omit it here, the idea of proof can be found in [4], Chapter 5.4., p.134.

Lemma 1.1. *For any symmetric non-negative bounded function f with bounded support and a locally-finite point measure N we have*

$$\int_{\mathbb{R}^n} f(\vec{x}) N^{\otimes n}(d\vec{x}) = \sum_{k=1}^{n} \sum_{\substack{1 \le l_1 \le \ldots \le l_k \\ l_1 + \ldots + l_k = n}} A^n_{l_1,\ldots,l_k} \int_{\mathbb{R}^k} f(\underbrace{x_1, \ldots, x_1}_{l_1}, \ldots, \underbrace{x_k, \ldots, x_k}_{l_k}) N^{(k)}(d\vec{x}),$$

$$\int_{\mathbb{R}^n} f(\vec{x}) N^{(n)}(d\vec{x}) = \sum_{k=1}^{n} \sum_{\substack{1 \le l_1 \le \ldots \le l_k \\ l_1 + \ldots + l_k = n}} a^n_{l_1,\ldots,l_k} \int_{\mathbb{R}^k} f(\underbrace{x_1, \ldots, x_1}_{l_1}, \ldots, \underbrace{x_k, \ldots, x_k}_{l_k}) N^{\otimes n}(d\vec{x}),$$

where the constants $A^n_{l_1,\ldots,l_k}$ and $a^n_{l_1,\ldots,l_k}$ do not depend on f and N.

Now we will define an integral with respect to the point measure N_t defined by (1.1), which corresponds to the Arratia flow at time t.

Theorem 1.1. *The operator A_t which is defined for $\phi \in C([0,1])$ as*

$$A_t\phi = \int_0^1 \phi(v) dN_t(v)$$

can be extended by continuity to a continuous operator acting from $L_2([0,1])$ to $L_2(\Omega, \mathcal{F}, \mathbf{P})$ and

$$\|A_t\|^2 \le \frac{1}{\pi t} + \frac{1}{\sqrt{\pi t}}.$$

Proof. For non-negative function ϕ, by Campbell formula [19]

$$\mathbf{E} A_t\phi = \int_0^1 \phi(v)\rho_t^{(1)}(v) dv$$

and by the definition of $\rho_t^{(1)}$ and $\rho_t^{(2)}$

$$\mathbf{E}\,(A_t\phi)^2 = \int_0^1 \int_0^1 \phi(v_1)\phi(v_2)\rho_t^{(2)}(v_1, v_2) dv_1\, dv_2 + \int_0^1 \phi^2(v_1)\rho_t^{(1)}(v_1) dv_1.$$

For the point measure N_t its $n-$point density functions $\rho_t^{(n)}$ satisfy inequality $\rho_t^{(n)} \leq \frac{1}{(\pi t)^{n/2}}$ for all $n \geq 1$, [21]. Using this, we can write

$$\mathbf{E}\,(A_t\phi)^2 \leq \left(\frac{1}{\pi t} + \frac{1}{\sqrt{\pi t}}\right) \|\phi\|^2_{L_2([0,1])}.$$

Using obtained inequality, the operator A_t can be extended from the operator defined on $C([0,1])$ to a continuous operator from $L_2([0,1])$ to $L_2(\Omega, \mathcal{F}, \mathbf{P})$. □

Remark 1.1. Note that the variance of the integral $A_t\phi = \int_0^1 \phi(v)dN_t(v)$ is given by the formula

$$\operatorname{Var} A_t\phi = \int_0^1 \int_0^1 \phi(v_1)\phi(v_2)\Big(\rho_t^{(2)}(v_1, v_2) - \frac{1}{\pi t}\Big)dv_1\, dv_2 + \frac{1}{\sqrt{\pi t}} \int_0^1 \phi^2(v_1)dv_1 \quad (1.4)$$

From Lemma 1.1, we get the formulas for moments of the defined integral.

Lemma 1.2. *Let $\phi \in C([0,1])$ then*

$$\mathbf{E}\left(\int_0^1 \phi(u)dN_t(u)\right)^k = \sum_{j=1}^{k} \sum_{\substack{l_1,\ldots,l_j \geq 1,\\ l_1+\ldots+l_j=k}} A^k_{l_1,\ldots,l_j} \int_{[0,1]^j} \prod_{i=1}^{j} \phi^{l_i}(x_i)\rho_t^{(j)}(\vec{x})d\,\vec{x},$$

with constants $A^k_{l_1,\ldots,l_j}$ from Lemma 1.1. In particular, for $k = 3$,

$$\mathbf{E}\left(\int_0^1 \phi(u)dN_t(u)\right)^3 = \int_{[0,1]^3} \phi^{\otimes 3}(\vec{u})\rho_t^{(3)}(\vec{u})d\vec{u} + 3\int_{[0,1]^2} \phi^{\otimes 2}(u_1, u_2)\rho_t^{(2)}(u_1, u_2)\varphi(u_1)d\vec{u} + \frac{1}{\sqrt{\pi t}} \int_{[0,1]} \phi^3(u)du.$$

Proof.

Since

$$\left(\int_0^1 \phi(u)dN_t(u)\right)^k = \int_{[0,1]^k} \phi^{\otimes k}(d\vec{u})N_t^{\otimes k}(d\vec{u}),$$

$\phi^{\otimes k}$ is a symmetric function and $\phi^{\otimes k} \in L^2([0,1]^k)$ than we can apply the equality from Lemma 1.1. □

Remark 1.2. Lemma 1.2 shows that $k-$th moments of the integral $\int_0^1 \phi(u)N_t(du)$ is finite if the integral $\int_0^1 |\phi(u)|^k du$ is finite.

Also, we can define a multiple integral with respect to the factorial power of the point measure N_t.

Theorem 1.2. *Let $L_{2,symm}([0,1]^k)$ be the space of square integrable symmetric functions. The linear operator A_t^k which is defined for symmetric function $\phi \in C([0,1]^k)$ as*

$$A_t^k\phi = \int_{[0,1]^k} \phi(\vec{v})N_t^{(k)}(d\,v_1, \ldots, d\,v_k),$$

can be extended by continuity to a continuous operator acting from $L_{2,symm}([0,1]^k)$ *to* $L_2(\Omega, \mathcal{F}, \mathbf{P})$.

Proof. By the definition of $k-$point density functions $\rho_t^{(k)}$ of the measure $N_t^{(k)}$

$$\mathbf{E}\int_{[0,1]^k}\phi(\vec{v})N_t^{(k)}(d\,v_1,\dots,d\,v_k) = \int_{[0,1]^k}\phi(\vec{v})\rho_t^{(k)}(\vec{v})d\,\vec{v}.$$

Rewriting the square of integral with respect to the factorial product as the sum of series indexed with non-equal indexes, we get

$$\begin{aligned}\mathbf{E}&\left(\int_{[0,1]^k}\phi(\vec{v})N_t^{(k)}(d\,v_1,\dots,d\,v_k)\right)^2\\ &= \mathbf{E}\sum_{i_1,j_1}\sum_{\substack{i_2\neq i_1\\ j_2\neq j_1}}\dots\sum_{\substack{i_k\neq i_1,\dots,i_{k-1}\\ j_k\neq j_1,\dots,j_{k-1}}}\phi(u_{i_1},\dots,u_{i_k})\phi(u_{j_1},\dots,u_{j_k})\\ &= \sum_{l=0}^{k}c_l\int_{[0,1]^k}\int_{[0,1]^{k-l}}\phi(\vec{y})\phi(y_1,\dots,y_l,v_1,\dots,v_{k-l})\rho^{(2k-l)}(\vec{v},\vec{y})d\,\vec{v}d\,\vec{y}\end{aligned}$$

with some constants $c_l \in \mathbb{N}$.

Applying uniform estimation for the $n-$point densities $\rho_t^{(n)} \leq \frac{1}{(\pi t)^{n/2}}$ [21] we get

$$\mathbf{E}\left(\int_{[0,1]^n}\phi(\vec{v})N_t^{(n)}(d\,v_1,\dots,d\,v_k)\right)^2 \leq C\int_{[0,1]^n}\phi^2(\vec{v})d\vec{v}.$$

This upper bound for the second moment proves the theorem. □

Remark 1.3. In the future we sometimes use the formula for A_t and $A_t^{(k)}$ as integral for function $\phi \in L_2([0,1]^k)$.

The next lemma states the shift-invariant property with respect to the spatial variable of the Arratia flow. Let us denote for any $h \in \mathbb{R}$ $x_h(u,\cdot) = x(u+h,\cdot) - h$.

Lemma 1.3. *For any real numbers* $n \in \mathbb{N}$, *any set of points* $\{u_1,\dots,u_n\}$ *and* $h \in \mathbb{R}$, *the following equality in the distribution in* $C([0,+\infty)^n)$ *holds*

$$(x(u_1,\cdot),\dots,x(u_n,\cdot)) \stackrel{d}{=} (x_h(u_1,\cdot),\dots,x_h(u_n,\cdot))$$

Proof. It is easy to see that for any $h \in \mathbb{R}$ the family of processes $\{x_h(u,\cdot),\ u \in \mathbb{R}\}$ satisfies all three properties from the definition of the Arratia flow. By [Theorem (86), 1] the distribution of such a system is unique. This proves the statement of the lemma. □

2. Finite-dimensional central limit theorem for $(\mathbf{A_t})_{\mathbf{t>0}}$

Let $\mathcal{P} = \{f : \mathbb{R} \to \mathbb{R} :\ f|_{[0,1]} \in L_2([0,1]),\ f(x) = f(x+1),\ x \in \mathbb{R}\}$ be a class of functions f such that:
1) f is 1-periodic;
2) restriction of f on the interval $[0,1]$ is square integrable.

Similarly to the definition of the operator A_t, let us define the integral operator

$$A_{k,t}f = \int_k^{k+1} f(u)N_t(du).$$

Such operators are well-defined for functions $f \in \mathcal{P}$. In this section, we will prove that the sequence $\{A_{k,t}f\}_{k\geq 1}$ satisfies the central limit theorem. The main reason why we can expect the validity of the central limit theorem is the weak dependence of the sequence $\{A_{k,t}f\}_{k\geq 1}$. It can be checked that this sequence satisfies a mixing condition.

Recall the definition of the mixing coefficient for a process.

Definition 2.1. For a random process $\{\xi_k\}_{k\in K}$ (here $K = \mathbb{R}$ or $K = \mathbb{N}$) its $\alpha-$mixing coefficient is defined as

$$\alpha_\xi(h) = \sup\{|\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)|,\ A \in \mathcal{F}(\xi)_{-\infty}^{u},\ B \in \mathcal{F}(\xi)_{u+h}^{\infty},\ u \in K\}$$

where $\mathcal{F}(\xi)_a^b = \sigma\{\xi_k,\ a < k < b\}$.

Note that for a stationary process $\{\xi_k\}_{k\in K}$,

$$\alpha_\xi(h) = \sup\{|\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)|,\ A \in \mathcal{F}(\xi)_{-\infty}^{0},\ B \in \mathcal{F}(\xi)_{h}^{\infty}\}$$

Lemma 2.1. *Let $f \in \mathcal{P}$. For fixed $t > 0$ consider the sequence $\xi_k = A_{k,t}f$, $k \in \mathbb{Z}$. Then*

$$\alpha_\xi(h) \leq 10\int_{h/2}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}dx \leq \frac{20\sqrt{t}}{h\sqrt{2\pi}} \exp(-h^2/(8t)).$$

Proof. For arbitrary $n, m \in \mathbb{N}$ denote by $D_{n,m}$ the set of vectors $\vec{u} \in \mathbb{R}^n$, $\vec{v} \in \mathbb{R}^m$, such that $u_1 < u_2 < \ldots u_n < v_1 < v_2 < \ldots < v_m$.

For arbitrary subsets $\Delta_1 \in \mathcal{B}(\mathbb{R}^n)$, $\Delta_2 \in \mathcal{B}(\mathbb{R}^m)$, let us denote

$$A_1(\vec{u}) = \{x(\vec{u},t) \in \Delta_1,\ x(u_n,t) < 0\},\ A_2(\vec{v}) = \{x(\vec{v},t) \in \Delta_2,\ x(v_1,t) > h\}$$

where we denoted by $x(\vec{u},t) = (x(u_1,t),\ldots,x(u_n,t))$. Let B be the random event $B = \{x(h/2,t) \in (0,h)\}$. From the definition of the Arratia flow the events A_1 and A_2 under the condition B are independent, i.e.

$$\mathbf{P}\left(A_1(\vec{u}), A_2(\vec{v})|B\right) = \mathbf{P}\left(A_1(\vec{u})|B\right)\mathbf{P}\left(A_2(\vec{v})|B\right).$$

From this, using notation $\overline{B}$ to denote the complement of the event B we get

$$\mathbf{P}(A_1(\vec{u})A_2(\vec{v})) = \mathbf{P}\left(A_1(\vec{u}) \mid B\right)\mathbf{P}\left(A_2(\vec{v}) \mid B\right)\mathbf{P}(B) + \mathbf{P}\left(A_1(\vec{u})A_2(\vec{v}) \mid \overline{B}\ \right)\mathbf{P}\left(\overline{B}\right)$$

and by simple calculation

$$\begin{aligned}&\mathbf{P}(A_1(\vec{u}))\mathbf{P}(A_2(\vec{v}))\\&=\mathbf{P}\left(A_1(\vec{u})\mid B\right)\mathbf{P}\left(A_2(\vec{u})\mid B\right)\mathbf{P}\left(B\right)\mathbf{P}\left(B\right)+\mathbf{P}\left(A_1(\vec{u})\mid B\right)\mathbf{P}\left(A_2(\vec{u})\mid \overline{B}\right)\mathbf{P}\left(B\right)\mathbf{P}\left(\overline{B}\right)\\&+\mathbf{P}\left(A_1(\vec{u})\mid \overline{B}\right)\mathbf{P}\left(A_2(\vec{u})\mid B\right)\mathbf{P}\left(\overline{B}\right)\mathbf{P}\left(B\right)+\mathbf{P}\left(A_1(\vec{u})\mid \overline{B}\right)\mathbf{P}\left(A_2(\vec{u})\mid \overline{B}\right)\mathbf{P}\left(\overline{B}\right)\mathbf{P}\left(\overline{B}\right)\end{aligned}$$

Now let us consider

$$\begin{aligned}\delta:=\sup_{D_{n,m}}&\left|\mathbf{P}(A_1(\vec{u})A_2(\vec{v}))-\mathbf{P}(A_1(\vec{u}))\ \mathbf{P}(A_2(\vec{v}))\right|\\&\le\mathbf{P}\left(A_1(\vec{u})\mid B\right)\mathbf{P}\left(A_2(\vec{v})\mid B\right)\mathbf{P}(B)\left(1-\mathbf{P}(B)\right)+4\mathbf{P}\left(\overline{B}\right)\\&\le 5(1-\mathbf{P}(B))\le 10\int_{h/2}^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}dx.\end{aligned}$$

Note that the upper bound does not depend on the choice of subsets $\Delta_1,\ \Delta_2$, so the statement of the lemma follows from obtained inequality since the family $\{\mathbb{1}_{\Delta}(x(\vec{u},t))\mathbb{1}_{\{x(u_n,t)<0\}},\ u_1<u_2<\ldots<u_n,\ n\in\mathbb{N},\Delta\in\mathcal{B}(\mathbb{R}^n)\}$ generates the $\sigma-$field $\sigma\{\xi_k=\int_k^{k+1}f(u)N_t(du),k<0\}$ and the family $\{\mathbb{1}_{\Delta}(x(\vec{v},t))\mathbb{1}_{\{x(v_1,t)>h\}},\ v_1<v_2<\ldots<v_n,\ n\in\mathbb{N},\Delta\in\mathcal{B}(\mathbb{R}^n)\}$ generates the $\sigma-$field $\sigma\{\xi_k=\int_k^{k+1}f(u)N_t(du),k>h\}$. □

Remark 2.1. Note that the proof remains the same for the sequence of random vectors $\{\vec{\xi}_k=(A_{k,t_1}f,\ldots,A_{k,t_m}f)\}_{k\in\mathbb{N}}$ with $t_1<\ldots<t_m$. The upper bound for the $\alpha-$mixing coefficient of $\vec{\xi}$ is

$$\alpha_{\vec{\xi}}(h)\le 10\int_{h/2}^{\infty}\frac{1}{\sqrt{2\pi t_m}}e^{-x^2/2t}dx\le\frac{20\sqrt{t_m}}{h\sqrt{2\pi}}\exp(-h^2/(8t_m)).$$

It is known that if the $\alpha-$mixing coefficient for a stationary sequence decreases fast enough, then under suitable moments conditions, this sequence satisfies the central limit theorem [16].

The next theorem gives the analog of the central limit theorem for the sequence $\{A_{k,t}f\}_{k\ge1}$. We use the notation '$\Rightarrow$' to denote weak convergence of a sequence of random variables.

Theorem 2.1.

(1) Let $f\in\mathcal{P}$ such that $f|_{[0,1]}\in L_3([0,1])$. For any $t>0$ we have

$$X_t^n(f)=\frac{\sum_{k=0}^{n-1}\left(A_{k,t}f-\mathbf{E}A_{k,t}f\right)}{\sqrt{n}}\Rightarrow\zeta_f(t),\ n\to\infty,$$

where $\zeta_f(t)$ is a Gaussian random variable with zero mean and the variance

$$\sigma_t^2(f)=\int_0^1\int_0^1 f(v_1)f(v_2)G_t(v_1,v_2)dv_1\,dv_2+\frac{1}{\sqrt{\pi t}}\int_0^1 f^2(v_1)dv_1,$$

with
$G_t(v_1, v_2) = g_t(v_1 - v_2) + 2\sum_{l=1}^{\infty} g_t(v_1 - v_2 + l)$,
and $g_t(v_1 - v_2) = \rho_t^{(2)}(v_1, v_2) - \frac{1}{\pi t}$.

(2) *For any* $0 < t_1 < t_2 < \ldots < t_m < T$ *and* $f \in \mathcal{P}$ *such that* $f|_{[0,1]} \in L_3([0,1])$

$$\left(X_{t_1}^n(f), \ldots, X_{t_m}^n(f)\right) \Rightarrow \left(\zeta_f(t_1), \ldots \zeta_f(t_m)\right), \ n \to \infty,$$

where $(\zeta_f(t_1), \ldots \zeta_f(t_m))$ *is a Gaussian vector with zero mean and the covariance matrix* $\Sigma_{t_1,\ldots,t_m}(f) = (c_{ij})_{i,j=1}^m$ *with*

$$c_{ij} = \lim_{n\to\infty} \frac{1}{n} \mathrm{cov}\left(X_{t_i}^n(f), X_{t_j}^n(f)\right), \ c_{ii} = \sigma_{t_i}^2(f),$$

(3) *For any functions* $f^{(1)}, \ldots, f^{(m)} \in \mathcal{P}$ *such that* $f^{(i)}|_{[0,1]} \in L_3([0,1])$ *and* $t > 0$ *we have*

$$\left(X_t^n\left((f^{(1)}\right), \ldots, X_t^n\left(f^{(m)}\right)\right) \Rightarrow \left(\zeta_{f^{(1)}}(t), \ldots, \zeta_{f^{(m)}}(t)\right), \ n \to \infty,$$

where $\left(\zeta_{f^{(1)}}(t), \ldots, \zeta_{f^{(m)}}(t)\right)$ *is a centered Gaussian vector with the covariance*

$$\begin{aligned}\mathrm{cov}\ &\left(\zeta_{f^{(i)}}(t), \zeta_{f^{(j)}}(t)\right) \\ &= \frac{1}{2}\int_0^1\int_0^1 \left[f^{(i)}(u)f^{(j)}(v) + f^{(i)}(v)f^{(j)}(u)\right] G_t(u,v)dudv \\ &\qquad + \frac{1}{\sqrt{\pi t}}\int_0^1 f^{(i)}(u)f^{(j)}(u)du.\end{aligned}$$

Remark 2.2. The expression for the covariances $c_{i,j}$ in the second statement of the theorem will be discussed in the next section.

Proof.

Proof of the first statement of the theorem.
For each $t > 0$ we denote

$$X_t^n(f) = \frac{\sum_{k=0}^{n-1}\left(A_{k,t}f - \mathbf{E}A_{k,t}f\right)}{\sqrt{n}} = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} Y_k.$$

Due to the periodicity of the function f and the stationarity with respect to the spatial variable of the Arratia flow, the sequence of random variables $\{Y_k\}_{k\in\mathbb{Z}}$ is strictly stationary. By Lemma 2.1, its mixing coefficient

$$\alpha_Y(h) \le c_t \int_{h/2}^{\infty} e^{-y^2/2t} dy.$$

By Lemma 1.2 $\mathbf{E}|Y_k|^3 < \infty$. Now we can apply the central limit theorem for a sequence of weakly dependent stationary random variables ([16], Theorem 18.5.3)

$$X_t^n(f) \Rightarrow \mathcal{N}(0, \sigma_t^2(f)), \text{ as } n \to \infty,$$

where

$$\sigma_t^2(f) = \lim_{n\to\infty} \frac{1}{n} \mathrm{Var} \sum_{k=0}^{n-1} Y_k$$
$$= \lim_{n\to\infty} \frac{1}{n} \int_0^n \int_0^n f(v_1) f(v_2) \Big(\rho_t^{(2)}(v_1, v_2) - \frac{1}{\pi t} \Big) dv_1\, dv_2 + \frac{1}{\sqrt{\pi t}} \int_0^n f^2(v_1) dv_1.$$

Note that the function $\Big(\rho_t^{(2)}(v_1, v_2) - \frac{1}{\pi t} \Big)$ depends on v_1, v_2 only via $|v_1 - v_2|$ (see formula (1.3)). Then we can define

$$g_t(v_1 - v_2) := \rho_t^{(2)}(v_1, v_2) - \frac{1}{\pi t}.$$

Using the periodicity of the function f

$$\sigma_t^2(f) = \lim_{n\to\infty} \frac{1}{n} \Big[\int_0^n \int_0^n f(v_1) f(v_2) g_t(v_1 - v_2) dv_1\, dv_2 + \frac{1}{\sqrt{\pi t}} \int_0^n f^2(v_1) dv_1 \Big]$$
$$= \lim_{n\to\infty} \frac{1}{n} \Big[\sum_{k_1,k_2=0}^{n-1} \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} f(v_1) f(v_2) g_t(v_1 - v_2) dv_1\, dv_2 + n \frac{1}{\sqrt{\pi t}} \int_0^1 f^2(v_1) dv_1 \Big]$$
$$= \lim_{n\to\infty} \frac{1}{n} \Big[\sum_{k_1,k_2=0}^{n-1} \int_0^1 \int_0^1 f(v_1) f(v_2) g_t(v_1 + k_1 - v_2 - k_2) dv_1\, dv_2 \Big] + \frac{1}{\sqrt{\pi t}} \int_0^1 f^2(v_1) dv_1$$
$$= \lim_{n\to\infty} \frac{1}{n} \int_0^1 \int_0^1 f(v_1) f(v_2) \Big[2 \sum_{l=1}^{n-1} (n-l) g_t(v_1 - v_2 + l) + n g_t(v_1 - v_2) \Big] dv_1\, dv_2$$
$$+ \frac{1}{\sqrt{\pi t}} \int_0^1 f^2(v_1) dv_1.$$

For the function $g_t(v_1 - v_2)$ one can write the precise formula using (1.3) :

$$g_t(v_1 - v_2) = \frac{1}{\pi t} \left(\frac{|v_2 - v_1|}{2\sqrt{t}} \cdot e^{-(v_2 - v_1)^2/4t} \cdot \int_{|v_2 - v_1|/\sqrt{t}}^{+\infty} e^{-v^2/4}\, dv - e^{-(v_2 - v_1)^2/2t} \right). \tag{2.1}$$

From this expression it is easy to see that the series $\sum_{l=1}^{\infty} l g_t(v_1 - v_2 + l)$ converges uniformly on $v_1, v_2 \in [0, 1]$, so we can pass to the limit with respect to n in the formula for $\sigma_t^2(f)$

$$\sigma_t^2(f) = \int_0^1 \int_0^1 f(v_1) f(v_2) \Big[g_t(v_1 - v_2) + 2 \sum_{l=1}^{\infty} g_t(v_1 - v_2 + l) \Big] dv_1\, dv_2$$
$$+ \frac{1}{\sqrt{\pi t}} \int_0^1 f^2(v_1) dv_1.$$

Note that we used the statement of the central limit theorem for weakly dependent stationary random variables with assumption $\sigma_t^2(f) > 0$. In the case when

$$\sigma_t^2(f) = \lim_{n\to\infty} \frac{1}{n} \mathrm{Var} \sum_{k=0}^{n-1} Y_k = 0,$$

the sequence $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} Y_k$ converges to 0 in L_2 and by distribution. We can interpret 0 as a Gaussian variable with zero mean and zero variance.

For the proof of the second statement we use the $\alpha-$mixing coefficient $\alpha(h)$ of the vector $(A_{k,t_1} f, \ldots, A_{k,t_m} f)$, with $0 < t_1 < \ldots < t_m < T$ that is given in the remark after Lemma 2.1.

Proof of the third statement of the theorem is similar to the first one. □

3. Central limit theorem for conditional expectations.

An attempt to find the covariances in the statement (2) of Theorem 2.1 naturally leads to the question of calculation conditional expectations $\mathbf{E}\left(\int_0^1 f(u) N_{t_2}(du) \big| \mathcal{F}_{t_1}^x\right)$. In this section, we study the limit behavior of the conditional expectations for $t_1 < t_2$

$$\frac{1}{\sqrt{n}} \mathbf{E}\left(\int_0^n f(u) N_{t_2}(du) \big| \mathcal{F}_{t_1}^x\right), \text{ as } n \to \infty,$$

where f is a 1-periodic function, such that $f \in C(\mathbb{R})$, $\int_0^1 f(u)du = 0$, N_t is the point measure associated to the Arratia flow x and $\mathcal{F}_t^x = \sigma\{x(u,s),\ u \in \mathbb{R},\ s \le t\}$.

To this aim, we need a representation of the Arratia flow in terms of the Brownian web. Recall that the Brownian web is defined as a family of random processes $\{\varphi_{s,\cdot}(u) \in C([s,\infty))\}_{(u,s)\in\mathbb{R}^2}$ such that, given $(u_1,t_1), ..., (u_m,t_m)$ the processes $\varphi_{t_1,\cdot}(u_1), ..., \varphi_{t_m,\cdot}(u_m)$ are coalescing Brownian motions (see, for instance, [12]). In this section, we consider the Arratia flow $x(u,t) = \varphi_{0,t}(u)$. We construct the point measure N_{t_2} using the measure N_{t_1} and the map φ_{t_1,t_2} from the Brownian web.

To describe such construction, let us introduce the family of random measures on $\mathbb{R}$ related to a point measure μ and a non-decreasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x\to+\infty} \varphi(x) = +\infty, \quad \lim_{x\to-\infty} \varphi(x) = -\infty.$$

For every $k \ge 2$ and a vector $\vec{v} = (v_1, ..., v_k)$ denote

$$v_* = \min\{v_1, ..., v_k\}, \quad v^* = \max\{v_1, ..., v_k\}.$$

Define the set

$$M_k^\varphi = \{\vec{v} : \varphi(v_*) = \varphi(v^*)\} \subset \mathbb{R}^k.$$

Note, that M_k^φ is a Borel subset of $\mathbb{R}^k$. Denote by $\mu_\varphi^{(k)}$ the restriction of the measure $\mu^{(k)}$ on the set M_k^φ. Finally, define the measure $\mu_{k,\varphi}$ as an image of the measure

$\mu_\varphi^{(k)}$ under the mapping

$$\mathbb{R}^k \ni \vec{v} = (v_1, \dots, v_k) \mapsto \varphi(v_*).$$

Note that $\mu_{1,\varphi} = \mu \circ \varphi^{-1}$. These measures we use to express the counting measure which is obtained from the measure μ under the action of a function φ. Define the counting measure $\nu = (\mu \circ \varphi^{-1})_*$ by

$$\nu(\Delta) = \#\{u \in \Delta : \ \mu(\varphi^{-1}\{u\}) > 0\}. \tag{3.1}$$

The integrals with respect to the measure ν can be expressed in the terms of the integrals with respect to $\mu_{k,\varphi},\ k \geq 1$.

Lemma 3.1. *Let f be a measurable bounded function on $\mathbb{R}$ with compact support and μ be a non-random locally-finite point measure. Consider the measure ν defined by (3.1) using the measure μ and a nondecreasing function φ such that*

$$\lim_{x\to+\infty} \varphi(x) = +\infty, \quad \lim_{x\to-\infty} \varphi(x) = -\infty.$$

Then the following relation holds

$$\int_{\mathbb{R}} f d\nu = \int_{\mathbb{R}} f d\mu_{1,\varphi} - \frac{1}{2!}\int_{\mathbb{R}} f d\mu_{2,\varphi} + \frac{1}{3!}\int_{\mathbb{R}} f d\mu_{3,\varphi} - \ldots + (-1)^{n+1}\frac{1}{n!}\int_{\mathbb{R}} f d\mu_{n,\varphi} + \ldots.$$

Proof. Since the measure μ is locally finite, and due to the condition on the function φ, the series from the right-hand side contains only a finite number of summands. To prove the lemma, it is enough to consider a point y such that $\nu(\{y\}) = 1$. Then for some ordered points $x_1 < \ldots < x_n$

$$\varphi(x_1) = \ldots = \varphi(x_n) = y, \quad \mu(\{x_1\}) = \ldots = \mu(\{x_n\}) = 1.$$

Consequently,

$$\nu(\{y\}) = \Big|\bigcup_{k=1}^n \{\varphi(x_k)\}\Big| = \sum_{k=1}^n |\{\varphi(x_k)\}| - \sum_{1\leq k_1<k_2\leq n} |\{\varphi(x_{k_1})\} \cap \{\varphi(x_{k_2})\}|$$
$$+ \ldots + (-1)^n \Big|\bigcap_{k=1}^n \{\varphi(x_k)\}\Big| = \mu_{1,\varphi}(\{y\}) - \frac{1}{2!}\mu_{2,\varphi}(\{y\}) + \ldots + \frac{1}{n!}\mu_{n,\varphi}(\{y\}).$$

This equality proves the lemma. □

Remark 3.1. The equality stated in Lemma 3.1 holds for the measure generated by the Arratia flow since it is a locally finite measure a.s.

Based on Lemma 3.1, we need to prove the limiting theorem for the integrals with respect to the measures $\mu_{k,\varphi}$ for the case when $\mu = N_{t_1}$ and φ_{t_1,t_2} is the mapping from the Brownian web. To calculate the corresponding mean and variance, let us find the expectation of the integrals with respect to measures $\mu_{k,\varphi}$. We will do this in two cases. One when μ is random and φ is deterministic and another when μ is deterministic and φ is random.

Lemma 3.2. *Suppose, that a random point measure μ has the point densities $\rho^{(k)}$, $k \geq 1$ and let φ be a deterministic non-decreasing function such that*

$$\lim_{x\to+\infty} \varphi(x) = +\infty, \quad \lim_{x\to-\infty} \varphi(x) = -\infty.$$

Then

$$\mathbf{E}\int_{\mathbb{R}} f d\mu_{k,\varphi} = \int_{\mathbb{R}^k} f(\varphi(v_*)) \mathbb{1}_{M_k^\varphi}(\vec{v}) \rho^{(k)}(\vec{v}) d\vec{v}$$

for any function f such that $\int_{\mathbb{R}^k} |f(\varphi(v_*))| \mathbb{1}_{M_k^\varphi}(\vec{v}) \rho^{(k)}(\vec{v}) d\vec{v} < \infty$.

Proof. The proof follows directly from the definition of measures $\mu_{k,\varphi}$. □

We need the following notation and assumption to formulate the statement when φ is random. Suppose that for arbitrary $a < b$ there exist a density $q(a, b, u)$ such that for every Borel subset $\Delta \subset \mathbb{R}$

$$\mathbf{P}\{\varphi(a) = \varphi(b),\ \varphi(a) \in \Delta\} = \int_\Delta q(a, b, u) du. \tag{3.2}$$

Lemma 3.3. *Let φ be a random non-decreasing function such that a.s.*

$$\lim_{x\to+\infty} \varphi(x) = +\infty, \quad \lim_{x\to-\infty} \varphi(x) = -\infty.$$

Assume also that φ has the density q defined by (3.2). Let μ be any deterministic point measure and f be a function such that $\mathbf{E}\int_{\mathbb{R}} |f| d\nu < \infty$, where the measure $\nu = (\mu \circ \varphi^{-1})_*$ is defined by (3.1). Then the following relation holds

$$\begin{aligned}\mathbf{E}\int_{\mathbb{R}} f d\nu = &\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(v) p(u, v) dv\right) \mu(du) \\ &- \frac{1}{2!}\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f(v) q(u_*, u^*, v) dv\right) \mu^{(2)}(d\vec{u}) \\ &+ \ldots + (-1)^{n+1} \frac{1}{n!}\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} f(v) q(u_*, u^*, v) dv\right) \mu^{(n)}(d\vec{u}) + \ldots.\end{aligned}$$

Here $p(u, \cdot)$ is the probability density of $\varphi(u)$, $u \in \mathbb{R}$.

Proof. The statement of the lemma follows from Lemma 3.1, the definitions of the measures $\mu_{k,\varphi}$ and the function q. □

Now we can consider summands from the series in the statement of Lemma 3.3 with a random stationary measure μ. To prove the convergence in distribution to a Gaussian random variables of integrals

$$\frac{1}{\sqrt{n}}\int_{\mathbb{R}^n} \left(\int_0^n f(v) q(u_*, u^*, v) dv\right) \mu^{(k)}(d\vec{u})$$

let us rewrite them. For every $k \geq 2$

$$\int_{\mathbb{R}^k} \left(\int_0^n f(v) q(u_*, u^*, v) dv\right) \mu^{(k)}(d\vec{u}) = \int_0^n f(v) \left(\int_{\mathbb{R}^k} q(u_*, u^*, v) \mu^{(k)}(d\vec{u})\right) dv.$$

Note that for a case when the random mapping φ_{t_1,t_2} is taken from the Brownian web, it can be easily checked that

(1) $\forall z \in \mathbb{R},\ a \le b,\ v \in \mathbb{R}$:

$$q_{t_2-t_1}(a,b,v) = q_{t_2-t_1}(a+z, b+z, v+z),$$

(2)

$$q_{t_2-t_1}(a,b,v) \le p_{t_2-t_1}(a,v) \wedge p_{t_2-t_1}(b,v),$$

where $p_t(a,\cdot)$ is the Gaussian density with mean a and variance t.

For the proof of the second property recall that we define q as the density of the following probability

$$\mathbf{P}\{\varphi(a) = \varphi(b),\ \varphi(a) \in \Delta\} = \int_\Delta q(a,b,u)du$$

Now, for a case when the random mapping φ_{t_1,t_2} is taken from the Brownian web we can write for every Δ

$$\mathbf{P}\{\varphi_{t_1,t_2}(a) = \varphi_{t_1,t_2}(b),\ \varphi_{t_1,t_2}(a) \in \Delta\} \le \mathbf{P}\{\ \varphi_{t_1,t_2}(a) \in \Delta\} \wedge \mathbf{P}\{\ \varphi_{t_1,t_2}(b) \in \Delta\}$$

From this follows the stated inequality for almost all v

$$q_{t_2-t_1}(a,b,v) \le p_{t_2-t_1}(a,v) \wedge p_{t_2-t_1}(b,v).$$

To see that this property is true for all v, we check the continuity of the function $q(a,b,\cdot)$. Let w_1, w_2 be two independent Wiener processes, such that $w_1(0) = a$, $w_2(0) = b$. Then for $a < b$

$$\mathbf{P}\{\varphi(a) = \varphi(b),\ \varphi(a) \in \Delta\} = \mathbf{P}\{\min_{r\in[0,t-s]}(w_2(r) - w_1(r)) \le 0,\ w_1(t-s) \in \Delta\}.$$

Define the function

$$G(x) = \mathbf{P}\{\min_{r\in[0,t-s]}(w_2(r) - w_1(r)) < 0,\ w_1(t-s) \le x\}.$$

Let us denote the set $D = \{x = (x_1, x_2) \in \mathbb{R}^2 :\ x_1 > x_2\}$ and by $p_{a,b,r}$ the density of 2-dimensional Wiener process that start form the point (a,b). Then using the reflection principle for the 2-dimensional Wiener process we get

$$\begin{aligned}
&G(x) = \mathbf{P}\{\exists r \in [0,t-s] :\ (w_1(r), w_2(r)) \in D,\ w_1(t-s) \le x\} \\
&= \mathbf{P}\{w_1(t-s) > w_2(t-s), w_1(t-s) \le x\} + \mathbf{P}\{w_1(t-s) > w_2(t-s), w_2(t-s) \le x\} \\
&\quad = \iint_{\mathbb{R}^2} p_{a,b,t-s}(u,v)\mathbb{1}_{\{v<u\le x\}}dudv + \iint_{\mathbb{R}^2} p_{a,b,t-s}(u,v)\mathbb{1}_{\{u\ge v, v\le x\}}dudv \\
&= \frac{2}{2\pi(t-s)}\int_{-\infty}^{x}\int_{-\infty}^{u} e^{-\frac{(a-u)^2+(b-v)^2}{2(t-s)}}dvdu + \frac{1}{2\pi(t-s)}\int_{x}^{+\infty}\int_{-\infty}^{x} e^{-\frac{(a-u)^2+(b-v)^2}{2(t-s)}}dvdu.
\end{aligned}$$

From this we get the continuity of the density $q(a,b,\cdot)$. Moreover, one can get precise formula, for $a<b$:

$$q_{t-s}(a,b,x)=\frac{1}{2\pi(t-s)}\left(e^{-\frac{(a-x)^2}{2(t-s)}}\int_{-\infty}^{x}e^{-\frac{(b-v)^2}{2(t-s)}}dv-e^{-\frac{(b-x)^2}{2(t-s)}}\int_{x}^{\infty}e^{-\frac{(a-v)^2}{2(t-s)}}dv\right).$$

Lemma 3.4. *Suppose that μ is a stationary random point measure. Then for every $k\geq 2$, the random function*

$$\xi_k(v)=\int_{\mathbb{R}^k}q_{t_2-t_1}(u_*,u^*,v)\mu^{(k)}(d\vec{u}),\ v\in\mathbb{R}$$

is a stationary random function .

Proof. Denote for arbitrary $z\in\mathbb{R}$ the image of the measure $\mu^{(k)}$ under the transformation

$$\mathbb{R}^k\ni(u_1,\dots,u_k)\mapsto(u_1+z,\dots,u_k+z)$$

by $\mu_z^{(k)}$. Since μ is a stationary random measure, then for every $z_1,\dots,z_n$ and $h\in\mathbb{R}$ the sequences of random measures

$$\left\{\mu_{z_1}^{(k)},\dots,\mu_{z_n}^{(k)}\right\},\left\{\mu_{z_1+h}^{(k)},\dots,\mu_{z_n+h}^{(k)}\right\}$$

are equidistributed. Now the statement of the lemma follows from the properties of the function $q_{t_2-t_1}$. □

Similarly to the mixing coefficients of the stationary process, we can define the mixing coefficient of a point measure.

Definition 3.1. A random stationary measure μ satisfies mixing condition with the function α if for any $t\in\mathbb{R}$ and $h>0$

$$\sup\left\{\mathbf{P}(A\cap B)-\mathbf{P}(A)\mathbf{P}(B)|,\ A\in\mathcal{F}_{-\infty}^{t},\ B\in\mathcal{F}_{t+h}^{\infty}\right\}\leq\alpha(h),$$

where $\mathcal{F}_s^t=\sigma\{\mu(\Delta):\Delta\subset[s,t]\}$.

Now we can prove the central limit theorem for the integrals with respect to the process ξ_k defined in Lemma 3.4.

Theorem 3.1. *Let the random mapping $\{\varphi_{t_1,t_2},\ t_1<t_2\}$ be the Brownian web and measure $\mu=N_{t_1}$ is the point measure that corresponds to φ_{0,t_1}. Let $f\in C(\mathbb{R})$ be a 1-periodic function such that $\int_0^1 f(u)du=0$. Then for every $k\in\mathbb{N}$ and $0<t_1<t_2<\infty$ there exists a Gaussian random variable $\zeta_k(f)$ such that*

$$\frac{1}{\sqrt{n}}\int_0^n f(v)\xi_k(v)dv\Rightarrow\zeta_k(f),\ n\to\infty,$$

where

$$\xi_k(v)=\int_{\mathbb{R}^k}q_{t_2-t_1}(u_*,u^*,v)\mu^{(k)}(d\vec{u}),\ v\in\mathbb{R}$$

with $q_{t_2-t_1}$ defined from

$$\mathbf{P}\{\varphi_{t_1,t_2}(a)=\varphi_{t_1,t_2}(b),\ \varphi_{t_1,t_2}(a)\in\Delta\}=\int_{\Delta}q_{t_2-t_1}(a,b,u)du.$$

Proof. Define the sequence of random variables

$$\eta_l=\int_l^{l+1}f(v)\xi_k(v)dv,\ l=0,1,\dots.$$

This sequence is stationary. Check that η_0 is centered.

$$\mathbf{E}\eta_0=\int_0^1 f(v)\mathbf{E}\xi_k(v)dv=\mathbf{E}\xi_k(0)\int_0^1 f(v)dv=0.$$

Now check the boundedness of the moments. Consider

$$\mathbf{E}\left|\int_0^1 f(v)\xi_k(v)dv\right|^n$$
$$\leq\int_0^1\dots\int_0^1|f(v_1)|\dots|f(v_n)|\mathbf{E}\prod_{j=1}^n\int_{\mathbb{R}^k}q_{t_k-t_1}(u_*,u^*,v_j)\mu^{(k)}(d\vec{u})d\vec{v}$$
$$\leq c_1\mathbf{E}\prod_{j=1}^n\int_{\mathbb{R}^k}e^{-c_1\|\vec{u}\|^2}\mu^{(k)}(d\vec{u}).$$

As in the proof of Theorem 1.2, using uniform estimation for the $n-$point densities for the point measure of the Arratia flow $\rho_s^{(n)}(\vec{u})\leq(\pi s)^{-n/2}$ (Lemma 2 in [21]), we can get the upper bound for $n-$th moment of the integral $\int_{\mathbb{R}^k}e^{-c_1\|\vec{u}\|^2}\mu^{(k)}(d\vec{u})$:

$$\mathbf{E}\left(\int_{\mathbb{R}^k}e^{-c_1\|\vec{u}\|^2}\mu^{(k)}(d\vec{u})\right)^n\leq C\int_{\mathbb{R}^k}e^{-c_1n\|\vec{u}\|^2}du.$$

From this boundedness of the $n-$th moment follows.

To establish the limit theorem, let us use approximations of the random processes ξ_k. For every $n\geq1$ define

$$\xi_k^m(v)=\int_{\mathbb{R}^k}q_{t_2-t_1}(u_*,u^*,v)\mathbb{1}_{\{v-m\leq u_*\leq u^*\leq v+m\}}\mu^{(k)}(d\vec{u}).$$

Similarly to the proof of Lemma 3.4 one can see that $\xi_k^m(\cdot)$ is a stationary random function. Note that the sequence of random variables

$$\eta_l^m=\int_l^{l+1}f(v)\xi_k^m(v)dv,\ l\geq0$$

is a sequence of centered random variables and satisfies mixing condition with the function $\alpha(\cdot-2m)$, where α is the function from mixing condition for measure μ. Due to the estimation of this function for the point measure from the Arratia flow, one can conclude that the following convergence holds

$$\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\eta_l^m\Rightarrow\varkappa_m,\quad n\to\infty.$$

Here $\varkappa_m$ is a centered Gaussian variable with the variance

$$\mathbf{E}\varkappa_m^2 = \int_0^1 \int_0^1 f(v_1)f(v_2)\mathbf{E}\prod_{j=1}^{2}\int_{\mathbb{R}^k} q_{t_2-t_1}(u_*,u^*,v_j)\mathbb{1}_{\{v_j-m\le u_*\le u^*\le v_j+m\}}\mu^{(k)}(d\vec{u})d\vec{v}.$$

For any $v\in\mathbb{R}$ and $m\ge 1$ denote

$$B_m(v)=\{\vec{u}\in\mathbb{R}^k:\ v-m\le u_*\le u^*\le u+m\}.$$

Then for every $m\ge 1$

$$\mathbf{E}\left(\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}(\eta_l-\eta_l^m)\right)^2 = \frac{1}{n}\mathbf{E}\left(\int_0^n\int_{\mathbb{R}^k\setminus B_m(v)} f(v)q_{t_2-t_1}(u_*,u^*,v)\mu^{(k)}(d\vec{u})dv\right)^2$$

$$= \frac{1}{n}\int_0^n\int_0^n f(v_1)f(v_2)\mathbf{E}\int_{\mathbb{R}^k\setminus B_m(v_1)}\int_{\mathbb{R}^k\setminus B_m(v_2)} q_{t_2-t_1}(u_{1*},u_1^*,v_1)q_{t_2-t_1}(u_{2*},u_2^*,v_2)$$
$$\cdot\,\mu^{(k)}(d\vec{u}_1)\mu^{(k)}(d\vec{u}_2)dv_1dv_2.$$

Consider expectation in the last expression. Using that

$$q_{t_2-t_1}(a,b,v)\le p_s(a,v)\wedge p_s(b,v),$$

where $p_t(a,\cdot)$ is the Gaussian density with mean a and variance t, and the uniform upper bound on the $k-$point densities of the point measure μ we get

$$\mathbf{E}\int_{\mathbb{R}^k\setminus B_m(v_1)}\int_{\mathbb{R}^k\setminus B_m(v_2)} q_{t_2-t_1}(u_{1*},u_1^*,v_1)q_{t_2-t_1}(u_{2*},u_2^*,v_2)\mu^{(k)}(d\vec{u}_1)\mu^{(k)}(d\vec{u}_2)$$
$$\le Ce^{-2\max(2m,|v_1-v_2|)}.$$

Substitute this upper bound into the expression for $\mathbf{E}\left(\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}(\eta_l-\eta_l^m)\right)^2$ we get

$$\mathbf{E}\left(\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}(\eta_l-\eta_l^m)\right)^2 \le C\frac{1}{n}\int_0^n\int_0^n e^{-\max(2m,|v_1-v_2|)}dv_1dv_2$$
$$= \frac{2C}{n}\int_0^n e^{-\max(2m,r)}dr\to 0,\ \text{as } m\to\infty.$$

From this we conclude that there exists σ^2 such that $\mathbf{E}\varkappa_m^2\to\sigma^2$ as $m\to\infty$ and

$$\varkappa_m\Rightarrow\varkappa,$$

where $\varkappa$ is a Gaussian random variable with zero mean and variance σ^2.

Denote the corresponding characteristic functions by $\varphi_n^m(\lambda)=\mathbf{E}\exp\left\{i\lambda\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\eta_l^m\right\}$, $\varphi_n(\lambda)=\mathbf{E}\exp\left\{i\lambda\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\eta_l\right\}$, $\varphi_\infty^m(\lambda)=\mathbf{E}\exp\{i\lambda\varkappa_m\}$ and $\varphi_\infty(\lambda)=\mathbf{E}\exp\{i\lambda\varkappa\}$. Since the weak convergence is equivalent

to point-wise convergence of characteristic functions, in this notation, we proved that

- $\varphi_\infty^m(\lambda) \to \varphi_\infty(\lambda)$ as $m \to \infty$, for all $\lambda \in \mathbb{R}$;
- for any fixed $m \geq 1$ $\varphi_n^m(\lambda) \to \varphi_\infty^m(\lambda)$ as $n \to \infty$, for all $\lambda \in \mathbb{R}$ and
- for any fixed $n \geq 1$ $\varphi_n^m(\lambda) \to \varphi_n(\lambda)$ as $m \to \infty$, for all $\lambda \in \mathbb{R}$.

Also, note that from the above inequality, we get

$$|\varphi_n(\lambda) - \varphi_n^m(\lambda)|^2 \leq \mathbf{E}\left(\frac{\lambda}{\sqrt{n}} \sum_{l=0}^{n-1} (\eta_l - \eta_l^m)\right)^2 \leq \frac{2C\lambda^2}{n} \int_0^n e^{-\max(2m,r)} dr.$$

Finally

$$|\varphi_n(\lambda) - \varphi_\infty(\lambda)| \leq |\varphi_n(\lambda) - \varphi_n^m(\lambda)| + |\varphi_n^m(\lambda) - \varphi_\infty^m(\lambda)| + |\varphi_\infty^m(\lambda) - \varphi_\infty(\lambda)|$$

For any given $\varepsilon > 0$ there exists such m_0 that $|\varphi_\infty^{m_0}(\lambda) - \varphi_\infty(\lambda)| < \varepsilon$. For this m_0 we can find n_0 such that $|\varphi_{n_0}(\lambda) - \varphi_{n_0}^{m_0}(\lambda)| + |\varphi_{n_0}^{m_0}(\lambda) - \varphi_\infty^{m_0}(\lambda)| < \varepsilon$. From this follows that $\varphi_n(\lambda) \to \varphi_\infty(\lambda)$ as $n \to \infty$ for all $\lambda \in \mathbb{R}$ and this proves the statement of the theorem. □

We can generalize this theorem to the multidimensional case.

Corollary 3.1. *Let φ_{t_1,t_2} be the Brownian web and measure $\mu = N_{t_1}$ is the point measure that corresponds to φ_{0,t_1}. Let $f_i \in C(\mathbb{R})$, $i = 1, \ldots, k$ be 1-periodic functions such that for $i = 1, \ldots, k$ $\int_0^1 f_i(u)du = 0$. Then there exists a Gaussian random vector $\vec{\zeta} := (\zeta_1(f_1), \zeta_2(f_2), \ldots, \zeta_k(f_k))$ such that*

$$\left(\frac{1}{\sqrt{n}} \int_0^n f_1(v)\xi_1(v)dv, \ldots, \frac{1}{\sqrt{n}} \int_0^n f_k(v)\xi_k(v)dv\right) \Rightarrow \vec{\zeta},\ n \to \infty,$$

where

$$\xi_k(v) = \int_{\mathbb{R}^k} q_{t_2-t_1}(u_*, u^*, v)\mu^{(k)}(d\vec{u}),\ v \in \mathbb{R}$$

with $q_{t_2-t_1}$ defined from

$$\mathbf{P}\{\varphi_{t_1,t_2}(a) = \varphi_{t_1,t_2}(b),\ \varphi_{t_1,t_2}(a) \in \Delta\} = \int_\Delta q_{t_2-t_1}(a, b, u)du.$$

Proof. The proof repeats all steps of the proof of the previous theorem. With the same notation as before, we note that the sequence of random vectors

$$\left(\int_l^{l+1} f_1(v)\xi_1^m(v)dv, \ldots, \int_l^{l+1} f_k(v)\xi_k^m(v)dv\right),\ l \geq 0$$

satisfies mixing condition with the function $\alpha(\cdot - 2m)$, where α is the function from mixing condition for measure μ.

For every $\lambda \in \mathbb{R}^k$ the sequence of linear combinations

$$\left\{ \sum_{j=1}^{k} \lambda_j \int_l^{l+1} f_1(v) \xi_1^m(v) dv \right\}_{l \geq 0}$$

converges in distribution to a centered Gaussian random variable. This means that the sequence of characteristic functions of the random vector converges to the characteristic function of the centered Gaussian distribution. From this the statement follows. □

Putting together Lemma 3.3 and Theorem 3.1 we get

Theorem 3.2. *Let $f \in C(\mathbb{R})$ be a 1-periodic function such that $\int_0^1 f(u)du = 0$ and $(N_t)_{t \geq 0}$ be the point measure that corresponds to the Arratia flow $\{x(u,t),\ u \in \mathbb{R},\ t \geq 0\}$. Then for $t_1 < t_2$*

$$\frac{1}{\sqrt{n}} \mathbf{E} \left(\int_0^n f(u) N_{t_2}(du) \big| \mathcal{F}_{t_1}^x \right) \Rightarrow \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k!} \zeta_k(f), \ \text{as } n \to \infty$$

where $\zeta_k(f)$ is a centered Gaussian variable with the variance

$$\mathbf{E}\zeta_k(f)^2 = \int_0^1 \int_0^1 \int_{\mathbb{R}^k} f(v_1) f(v_2) \prod_{j=1}^{2} q_{t_2 - t_1}(u_*, u^*, v_j) \rho_{t_1}^{(k)}(\vec{u})(d\vec{u}) dv_1 dv_2.$$

Proof. Lemma 3.3 states that for a deterministic measure μ and a random function φ the following relation holds

$$\begin{aligned} \mathbf{E} \int_{\mathbb{R}} f d\nu = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(v) p(u,v) dv \right) \mu(du) \\ - \frac{1}{2!} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f(v) q(u_*, u^*, v) dv \right) \mu^{(2)}(d\vec{u}) \\ + \ldots + (-1)^{n+1} \frac{1}{n!} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} f(v) q(u_*, u^*, v) dv \right) \mu^{(n)}(d\vec{u}) + \ldots . \end{aligned}$$

Now for $t_1 < t_2$ let us take $\mu = N_{t_1}$ and $\varphi = \varphi_{t_1,t_2}$, where φ_{t_1,t_2} is the Brownian web. Then $N_{t_2} = (\mu \circ \varphi^{-1})_*$ as it was defined in the equation (3.1). Since now μ and φ are independent, from Lemma 3.3, we get

$$\begin{aligned} \frac{1}{\sqrt{n}} \mathbf{E} \left(\int_0^n f(u) N_{t_2}(du) | \mathcal{F}_{t_1}^x \right) \\ = \frac{1}{\sqrt{n}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \int_{\mathbb{R}^k} \left(\int_0^n f(v) q_{t_2 - t_1}(u_*, u^*, v) dv \right) N_{t_1}^{(k)}(d\vec{u}), \end{aligned}$$

where $q_{t_2 - t_1}$ is defined trough the equality

$$\mathbf{P}\{\varphi_{t_1,t_2}(a) = \varphi_{t_1,t_2}(b),\ \varphi_{t_1,t_2}(a) \in \Delta\} = \int_{\Delta} q_{t_2 - t_1}(a,b,u) du.$$

By the Corollary of Theorem 3.1, the partial sum weakly converges as $n \to \infty$

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{N} \frac{(-1)^{k+1}}{k!} \int_{\mathbb{R}^k} \left(\int_0^n f(v) q_{t_2-t_1}(u_*, u^*, v) dv \right) N_{t_1}^{(k)}(d\vec{u}) \Rightarrow \sum_{k=1}^{N} \frac{(-1)^{k+1}}{k!} \zeta_k(f).$$

In the proof of the Theorem 3.1, we get that the variance of $\zeta_k(f)$ is equal to

$$\mathbf{E}(\zeta_k(f))^2 = \lim_{m\to\infty} \int_0^1 \int_0^1 f(v_1) f(v_2) \mathbf{E} \prod_{j=1}^{2} \int_{\mathbb{R}^k} q_{t_2-t_1}(u_*, u^*, v_j) \mathbb{1}_{\{v_j - m \le u_* \le u^* \le v_j + m\}} N_{t_1}^{(k)}(d\vec{u}) d\vec{v}.$$

Recall that

$$q_{t_2-t_1}(a, b, v) \le p_{t_2-t_1}(a, v) \wedge p_{t_2-t_1}(b, v),$$

where $p_t(a, \cdot)$ is the Gaussian density with mean a and variance t. Using that the $n-$point density of the measure N_{t_1} is bounded uniformly by $\rho_{t_1}^{(k)} \le (\pi t_1)^{-k/2}$ we get

$$\mathbf{E}(\zeta_k(f))^2 \le \int_0^1 \int_0^1 |f(v_1) f(v_2)| \mathbf{E} \prod_{j=1}^{2} \int_{\mathbb{R}^k} q_{t_2-t_1}(u_*, u^*, v_j) N_{t_1}^{(k)}(d\vec{u}) d\vec{v} \le (\pi t_1)^{-k/2} \int_0^1 f(v)^2 dv \left(\int_{\mathbb{R}^k} \frac{e^{-\|\vec{u}\|^2/(t_2-t_1)}}{(\pi(t_2-t_1))^{k/2}} d\vec{u} \right)^2.$$

So finally, we get

$$\mathbf{E}(\zeta_k(f))^2 \le (\pi t_1)^{-k/2} \int_0^1 f(v)^2 dv.$$

Using this estimation, we get

$$\lim_{N\to\infty} \mathbf{E} \left(\sum_{k=1}^{N} (-1)^{k+1} \frac{1}{k!} \zeta_k(f) \right)^2 \le \lim_{N\to\infty} \left(\sum_{k=1}^{N} \frac{1}{k!} \right) \left(\mathbf{E} \sum_{k=1}^{N} \frac{\zeta_k^2(f)}{k!} \right) < \infty.$$

From this follows that the sequence of centered Gaussian random variables

$$\left\{ \sum_{k=1}^{N} (-1)^{k+1} \frac{1}{k!} \zeta_k(f) \right\}_{N \ge 1}$$

weakly converges to a centered Gaussian variable

$$\zeta(f) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k!} \zeta_k(f).$$

Note that the series

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{N}\frac{(-1)^{k+1}}{k!}\int_{\mathbb{R}^k}\left(\int_0^n f(v)q_{t_2-t_1}(u_*,u^*,v)dv\right)N_{t_1}^{(k)}(d\vec{u})$$

converges as $N\to\infty$ in L_2 uniformly with respect to n. From this, the statement of the theorem can be obtained similarly to the proof of Theorem 3.1. □

4. Convergence of integrals in functional spaces.

Note that the second part of the Theorem 2.1 states that for a fixed function f and a finite set of moments of time, there exists the limiting Gaussian random vector of the sequence of the integrals. Moreover, the family of these distributions indexed by a set of moments of time is consistent. Hence, by the Kolmogorov theorem, a Gaussian process exists with such finite-dimensional distributions. This process is defined by its covariance function from Theorem 2.1. In this section, we consider the question about the convergence of the sequence of processes $\{X_t^n(f)=\frac{1}{\sqrt{n}}\int_0^n f(u)N_t(du),\ t\in[t_0,T]\}_{n\geq 1}$.

First of all, we prove that for a continuous function with bounded support, the random process, which is defined as integral $\{\int_{\mathbb{R}} f(u)N_t(du),\ t\in[t_0,T]\}$ has right-continuous modification with left limit. For this, let us put $D_n=\{\frac{k}{2^n},\ k\in\mathbb{Z}\cap[-n2^n,n2^n]\}$, and $\sigma_k^{(n)}=\inf\{t:\ x\left(\frac{k}{2^n},t\right)=x\left(\frac{k-1}{2^n},t\right)\}\wedge T$, $k\in\mathbb{Z}\cap[-n2^n,n2^n]$, $n\in\mathbb{N}$.

Theorem 4.1. *For any continuous function with bounded support $f\in C_0(\mathbb{R})$ and for any $t_0>0$ the function $\{N_t(f)\}_{t\in[t_0,T]}$ is right continuous with left limits and for any finite signed measure $\varkappa$ on $[t_0,T]$*

$$\int_{t_0}^T N_t(f)\varkappa(dt)=\int_{\mathbb{R}}\int_{t_0}^{(T\wedge\sigma(u))\vee t_0}f(x(u,t))\varkappa(dt)$$
$$:=\lim_{n\to\infty}\sum_{k=-n2^n}^{n2^n}\int_{t_0}^{\left(T\wedge\sigma_k^{(n)}\right)\vee t_0}f\left(x\left(\frac{k}{2^n},t\right)\right)\varkappa(dt).$$

Proof. Let us consider the case of a nonnegative function f and a positive measure $\varkappa$. For the set $D_n=\{\frac{k}{2^n},\ k\in\mathbb{Z}\cap[-n2^n,n2^n]\}$ define the corresponding point process

$$N_t^{D_n}(\Delta)=\#\{x(D_n,t)\cap\Delta\}.$$

Then it is easy to see that

$$N_t^{D_n}(f)=\int_{\mathbb{R}}f(u)N_t^{D_n}(du)=\sum_{k=-n2^n}^{n2^n}f\left(x\left(\frac{k}{2^n},t\right)\right)\mathbb{1}_{\{t<\sigma_k^{(n)}\}}.$$

From this representation, it is clear that the function $N_t^{D_n}(f)$ is right-continuous with left limits. The sequence

$$\left\{S_n = N_t^{D_n}(f)\right\}_{n\geq 1}$$

is monotone. Indeed, let us consider points

$$\left\{\frac{2k}{2^{n+1}},\ \frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right\} \in D_{n+1}$$

and compare two sums

$$\sum_{j=k}^{k+1} f\left(x\left(\frac{j}{2^n}, t\right)\right) 1\!\!1_{\left\{t<\sigma_j^{(n)}\right\}} \text{ and } \sum_{j=2k}^{2k+2} f\left(x\left(\frac{j}{2^{n+1}}, t\right)\right) 1\!\!1_{\left\{t<\sigma_j^{(n+1)}\right\}}$$

In the case when the trajectory $x\left(\frac{2k+1}{2^{n+1}}, \cdot\right)$ meets the trajectory $x\left(\frac{2k}{2^{n+1}}, \cdot\right)$ before $x\left(\frac{2k+2}{2^{n+1}}, \cdot\right)$ (this means that $\sigma_{2k+1}^{(n)} < \sigma_{2k+2}^{(n)}$), these two sums differ by one nonnegative value $f\left(x\left(\frac{2k+1}{2^n}, t\right)\right) 1\!\!1_{\left\{t<\sigma_{2k+1}^{(n+1)}\right\}}$. In the case when the trajectory $x\left(\frac{2k+1}{2^{n+1}}, \cdot\right)$ meets the trajectory $x\left(\frac{2k+2}{2^{n+1}}, \cdot\right)$ before $x\left(\frac{2k}{2^{n+1}}, \cdot\right)$ (this means that $\sigma_{2k+1}^{(n)} > \sigma_{2k+2}^{(n)}$) these two sums differ by one nonnegative value $f\left(x\left(\frac{2k+1}{2^{n+1}}, t\right)\right) 1\!\!1_{\left\{t<\sigma_{2k+2}^{(n+1)}\right\}}$. Consequently, the sequence

$$\int_{\mathbb{R}} f(u) N_t^{D_n}(du) = \sum_{k=-n2^n}^{n2^n} f\left(x\left(\frac{k}{2^n}, t\right)\right) 1\!\!1_{\{t<\sigma_k^{(n)}\}}$$

is monotone. Now let us consider the integral $\int_{t_0}^T \int_{\mathbb{R}} f(u) N_t^{D_n}(du) \varkappa(dt)$ and show that there exists a constant $K > 0$ such that

$$\mathbf{E} \sum_{k=-n2^n}^{n2^n} \int_{t_0}^{\left(T\wedge\sigma_k^{(n)}\right)\vee t_0} f\left(x\left(\frac{k}{2^n}, t\right)\right) \varkappa(dt) \leq K.$$

Note that

$$\mathbf{E} \int_{t_0}^T 1\!\!1_{\{t_0<t<\sigma_k^{(n)}\vee t_0\}} \varkappa(dt) \leq C \cdot \frac{1}{2^n},$$

where the constant C depends on t_0 and the measure $\varkappa$. Indeed, let us denote for a standard Wiener process w, the moment of time

$$\tau_u = \inf\{t:\ w(t) = u\} \wedge 1.$$

Then

$$\int_{t_0}^T \mathbf{P}\{\tau_u \geq t\} \varkappa(dt) = \frac{1}{\sqrt{2\pi}} \int_{t_0}^T \int_{-u}^u \frac{1}{\sqrt{t}} e^{-v^2/(2t)} dv \varkappa(dt)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-u}^u \int_{t_0}^T \frac{1}{\sqrt{t}} e^{-v^2/(2t)} \varkappa(dt) dv \leq \frac{1}{\sqrt{2\pi}} \int_{-u}^u \int_{t_0}^T \frac{1}{\sqrt{t}} \varkappa(dt) dv = \frac{2u}{\sqrt{2\pi t_0}} \varkappa([t_0, T])$$

Since f has bounded support, there exists such $M > 0$ that $\operatorname{supp} f \subset [-M, M]$. Let us consider the expectation of $\int_{t_0}^{\left(T\wedge\sigma_k^{(n)}\right)\vee t_0} f\left(x\left(\sigma_k^{(n)}, t\right)\right) \varkappa(dt)$. Denoting by w the standard Brownian motion, for $|k| > M \cdot 2^n + 1$ we have

$$\begin{aligned}
&\mathbf{E}\int_{t_0}^{T\wedge\sigma_k^{(n)}\vee t_0} f\left(x\left(\frac{k}{2^n}, t\right)\right) \varkappa(dt) \\
&\qquad \le \|f\| \cdot \mathbf{E}\left[\int_{t_0}^{T} \mathbb{1}_{\{t_0<t<\sigma_k^{(n)}\vee t_0\}} \varkappa(dt) \mathbb{1}_{\left\{x\left(\frac{k}{2^n},[0,T]\right)\cap[-M,M]\neq\varnothing\right\}}\right] \\
&\le \|f\| \cdot \left(\mathbf{E}\int_{t_0}^{T} \mathbb{1}_{\{t_0<t<\sigma_k^{(n)}\vee t_0\}} \varkappa(dt)\right)^{1/2} \left(\mathbf{P}\left\{x\left(\frac{k}{2^n},[0,T]\right)\cap[-M,M]\neq\varnothing\right\}\right)^{1/2} \\
&\qquad \le \|f\| \left(\frac{C}{2^n}\right)^{1/2} \left(\mathbf{P}\left\{\max_{s\in[0,T]} w(s) > \frac{|k|}{2^n} - M\right\}\right)^{1/2} \\
&\qquad \le c\|f\| \left(\frac{1}{2^{n/2}}\right) e^{-\left(\frac{|k|}{2^n}-M\right)^2/4T},
\end{aligned}$$

where the constant c depends on t_0 and $\varkappa$. For any k, we can write:

$$\mathbf{E}\int_{t_0}^{T\wedge\sigma_k^{(n)}} f\left(x\left(\frac{k}{2^n}, t\right)\right) \varkappa(dt) \le \|f\| \cdot \mathbf{E}\int_{t_0}^{T} \mathbb{1}_{\{t_0<t<\sigma_k^{(n)}\vee t_0\}} \varkappa(dt) \le c\|f\|\frac{1}{2^n}.$$

Note that using the same argument we can get for $|k| > M \cdot 2^n + 1$

$$\mathbf{E}\sup_{t_0\le t\le T} f\left(x\left(\frac{k}{2^n}, t\right)\right) \mathbb{1}_{\{t<\sigma_k^{(n)}\}} \le c\|f\| \left(\frac{1}{2^{n/2}}\right) e^{-\left(\frac{|k|}{2^n}-M\right)^2/(2T)}$$

and for any k,

$$\mathbf{E}\sup_{t_0\le t\le T} f\left(x\left(\frac{k}{2^n}, t\right)\right) \mathbb{1}_{\{t<\sigma_k^{(n)}\}} \le c\|f\|\frac{1}{2^n}.$$

From this, we get

$$\begin{aligned}
&\sum_{k=0}^{n2^n} \mathbf{E}\sup_{t_0\le t\le T} f\left(x\left(\frac{k}{2^n}, t\right)\right) \mathbb{1}_{\{t<\sigma_k^{(n)}\}} \\
&\qquad \le \sum_{k=0}^{M2^n} \mathbf{E}\sup_{t_0\le t\le T} f\left(x\left(\frac{k}{2^n}, t\right)\right) \mathbb{1}_{\{t<\sigma_k^{(n)}\}} \\
&\qquad + \sum_{k=M2^n+1}^{\infty} \mathbf{E}\sup_{t_0\le t\le T} f\left(x\left(\frac{k}{2^n}, t\right)\right) \mathbb{1}_{\{t<\sigma_k^{(n)}\}} \\
&\qquad \le \|f\| \left(\frac{1}{2^n} M2^n + \sum_{k=1}^{\infty} e^{-k^2/(2T)}\right) \le C\|f\|,
\end{aligned}$$

and so for all $n \ge 1$

$$\mathbf{E}\sup_{t_0\le t\le T} N_t^{D_n}(f) \le C\|f\|.$$

To check that $\{N_t(f)\}_{t\in[t_0,T]}$ is right-continuous, we consider the right-continuous modification of $x(\cdot,t_0)$ and so for any $\omega\in\Omega$, there exists $n=n(\omega)$ such that

$$x(\mathbb{R},t_0)\cap[-M,M]=x(D_{n(\omega)},t_0)\cap[-M,M].$$

It follows from this that for $t>t_0$

$$N_t(f)(\omega)=N_t^{D_{n(\omega)}}(f)(\omega),$$

therefore, $N_t(f)(\omega)$ is right-continuous with left limits on $[t_0,T]$.

To check the representation for $\int_{t_0}^T N_t(f)\varkappa(dt)$ given in the statement of the theorem, we use the upper bounds for $\mathbf{E}\int_{t_0}^{T\wedge\sigma_k^{(n)}} f\left(x\left(\frac{k}{2^n},t\right)\right)\varkappa(dt)$ to get

$$\begin{aligned}&\sum_{k=0}^{n2^n}\mathbf{E}\int_{t_0}^{T\wedge\sigma_k^{(n)}} f\left(x\left(\frac{k}{2^n},t\right)\right)\varkappa(dt)\\ &\le\sum_{k=0}^{M2^n}\mathbf{E}\int_{t_0}^{T\wedge\sigma_k^{(n)}} f\left(x\left(\frac{k}{2^n},t\right)\right)\varkappa(dt)+\sum_{k=M2^n+1}^{\infty}\mathbf{E}\int_{t_0}^{T\wedge\sigma_k^{(n)}} f\left(x\left(\frac{k}{2^n},t\right)\right)\varkappa(dt)\\ &\le|\varkappa|\|f\|\left(\frac{1}{2^n}M2^n+\sum_{k=1}^{\infty}e^{-k^2/(2T)}\right)\le C|\varkappa|\|f\|.\end{aligned}$$

So the sequence $\int_{t_0}^T\int_{\mathbb{R}} f(u)N_t^{D_n}(du)$ is monotone and bounded that allows as to pass to the limit and get

$$\int_{t_0}^T N_t(f)\varkappa(dt)=\lim_{n\to\infty}\sum_{k=-n2^n}^{n2^n}\int_{t_0}^{\left(T\wedge\sigma_k^{(n)}\right)\vee t_0} f\left(x\left(\frac{k}{2^n},t\right)\right)\varkappa(dt)\qquad\square$$

Remark 4.1. The similar construction of the integrals up to the first meeting time of trajectories was proposed by A.A. Dorogovtsev in [7] when he investigated the total time of free motion of particles in the Arratia flow and his construction of the stochastic integral with respect to the Arratia flow. Also, this limiting procedure was used in [3] when the author investigated local time at zero of Arratia flow. We can use the notation which was proposed in mentioned papers:

$$\begin{aligned}&\int_{\mathbb{R}}\int_{t_0}^{(T\wedge\sigma(u))\vee t_0} f(x(u,t))\varkappa(dt)\\ &\qquad:=\lim_{n\to\infty}\sum_{k=-n2^n}^{n2^n}\int_{t_0}^{\left(T\wedge\sigma_k^{(n)}\right)\vee t_0} f\left(x\left(\frac{k}{2^n},t\right)\right)\varkappa(dt)\end{aligned}$$

Corollary 4.1. *For any finite signed measure $\varkappa$, the integral*

$$\nu(\cdot)=\int_{t_0}^T N_t(\cdot)\varkappa(dt)$$

is well defined and ν is a signed measure.

Proof. By the previous theorem, for a finite signed measure $\varkappa$ and a continuous function with bounded support $f \in C_0(\mathbb{R})$ the integral

$$\Phi(f) := \int_{t_0}^{T} N_t(f)\varkappa(dt)$$

is well-defined. Without loss of generality, we can consider a measure $\varkappa$ (nonnegative). In this case the linear functional $\Phi : C_0(\mathbb{R}) \to \mathbb{R}$ is nonnegative and bounded:

$$|\Phi(f)| \le \|f\| \int_{t_0}^{T} N_t([-M, M])\varkappa(dt),$$

where $M > 0$ is a such constant that $\operatorname{supp} f \in [-M, M]$. By Riesz representation theorem, there exists a measure $N(\varkappa)$ such that

$$\int_{t_0}^{T} N_t(f)\varkappa(dt) = \int_{\mathbb{R}} f(u)N(\varkappa)(du).$$

From this follows that the measure $\int_{t_0}^{T} N_t(\cdot)\varkappa(dt)$ is well defined. □

In the next theorem, we state the conditions on a function f, under which the Gaussian process defined by the finite-dimensional distributions from the Theorem 2.1 has continuous modification.

Theorem 4.2. *Let $f \in C(\mathbb{R})$ be a 1-periodic Lipshitz continuous function such that $\int_0^1 f(u)du = 0$. Then the limiting Gaussian process $X_t(f), t \in [t_0, T]$ with finite-dimensional distributions from the Theorem 2.1 has continuous modification.*

Proof. Let the random mapping $\{\varphi_{t_1,t_2},\ t_1 < t_2\}$ be the Brownian web and the measure N_t is the point measure that corresponds to $\varphi_{0,t}$. For $s < t$ we consider

$$\mathbf{E}\left(\frac{1}{\sqrt{n}}\left(\int_0^n f(u)N_s(du) - \int_0^n f(u)N_t(du)\right)^2\right.$$
$$\left. = \frac{1}{n}\mathrm{Var}\left(\int_0^n f(u)N_s(du) - \int_0^n f(u)N_t(du)\right)^2.\right.$$

For $s > 0$ we enumerate points of the set $\{\varphi_{0,s}(u), u \in \mathbb{R}\}$ by increasing order and denote them as $\{\xi_{s,i}\}_{i\in\mathbb{Z}}$ (by this definition, $\xi_{s,i} < \xi_{s,j},\ i < j$). Let us denote by

$$I_{s,t} = \{i \in \mathbb{Z} :\ \forall j < i \quad \varphi_{s,t}(\xi_{s,i}) \neq \varphi_{s,t}(\xi_{s,j})\}.$$

Note that for $i_1, i_2 \in I_{s,t}\quad \varphi_{s,t}(\xi_{s,i_1}) \neq \varphi_{s,t}(\xi_{s,i_2})$, and $\varphi_{s,t}\left(\{\xi_{s,i}\}_{i\in I_{s,t}}\right) = \{\xi_{t,i}\}_{i\in\mathbb{Z}}$. Now we can represent the integral $\int_0^n f(u)N_t(du)$ in terms of the points $\{\xi_{s,i}\}_{i\in\mathbb{Z}}$ and mapping $\{\varphi_{s,t}\}$:

$$\int_0^n f(u)N_t(du) = \sum_{i\in\mathbb{Z}} f(\xi_{t,i})\,\mathbb{1}_{\{\xi_{t,i}\in[0,n]\}} = \sum_{i\in I_{s,t}} f(\varphi_{s,t}(\xi_{s,i}))\,\mathbb{1}_{\{\varphi_{s,t}(\xi_{s,i})\in[0,n]\}}$$

$$= \sum_{i\in I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1\!\!1_{\{\xi_{s,i}\in[0,n])} + \sum_{i\in I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1_{\{\xi_{s,i}\in\mathbb{R}\setminus[0,n],\varphi(\xi_{s,i})\in[0,n]]}$$

$$- \sum_{i\in I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1\!\!1_{\{\xi_{s,i}\in[0,n],\varphi(\xi_{s,i})\notin[0,n]\}}$$

$$= \sum_{i\in\mathbb{Z}} f\left(\varphi_{s,i}\left(\xi_{s,i}\right)\right) 1\!\!1_{\{\xi_{s,i}\in[0,n]\}} - \sum_{i\in\mathbb{Z}\setminus I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1\!\!1_{\{\xi_{s,i}\in[0,n]\}}$$

$$+ \sum_{i\in I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1\!\!1_{\{\xi_{s,i}\in\mathbb{R}\setminus[0,n],\varphi(\xi_{s,i})\in[0,n]\}} - \sum_{i\in I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1\!\!1_{\{\xi_{s,i}\in[0,n],\varphi(\xi_{s,i})\notin[0,n]\}}.$$

Then

$$\int_0^n f(u)N_s(du) - \int_0^n f(u)N_t(dus)$$

$$= \left(\sum_{i\in\mathbb{Z}} f\left(\xi_{s,i}\right) 1\!\!1_{\{\xi_{s,i}\in[0,n]\}} - \sum_{i\in\mathbb{Z}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1\!\!1_{\{\xi_{s,i}\in[0,n]\}}\right)$$

$$+ \sum_{i\in\mathbb{Z}\setminus I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1\!\!1_{\{\xi_{s,i}\in[0,n]\}}$$

$$- \sum_{i\in I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1_{\{\xi_{s,i}\in\mathbb{R}\setminus[0,n],\varphi(\xi_{s,i})\in[0,n]\}} + \sum_{i\in I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1\!\!1_{\{\xi_{s,i}\in[0,n],\varphi(\xi_{s,i})\notin[0,n]\}}.$$

Now we use $\operatorname{Var}\left(X_1+X_2\right) = \operatorname{Var}\left(X_1\right) + 2\operatorname{Cov}\left(X_1,X_2\right) + \operatorname{Var}\left(X_2\right)$

$$\leq \operatorname{Var}\left(X_1\right) + 2\sqrt{\operatorname{Var} X_1 \operatorname{Var} X_2} + \operatorname{Var}\left(X_2\right) \leq 2\operatorname{Var}\left(X_1\right) + 2\operatorname{Var}\left(X_2\right)$$

and get

$$\operatorname{Var}\left(\int_0^n f(u)N_s(du) - \int_0^n f(u)N_t(du)\right) \leq 4\left(V_1+V_2+V_3+V_4\right),$$

where

$$V_1 = \operatorname{Var}\left(\int_0^n \left[f(u) - f\left(\varphi_{s,t}(u)\right)\right] N_s(du)\right), \tag{4.1}$$

$$V_2 = \operatorname{Var}\left(\sum_{i\in\mathbb{Z}\setminus I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1\!\!1_{\{\xi_{s,i}\in[0,n])}\right), \tag{4.2}$$

$$V_3 = \operatorname{Var}\left(\sum_{i\in I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1\!\!1_{\{\xi_{s,i}\in\mathbb{R}\setminus[0,n],\varphi_{s,t}(\xi_{s,i})\in[0,n]\}}\right), \tag{4.3}$$

$$V_4 = \operatorname{Var}\left(\sum_{i\in I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) 1\!\!1_{\{\xi_{s,i}\in[0,n],\varphi_{s,t}(\xi_{s,i})\notin[0,n]\}}\right). \tag{4.4}$$

Let us consider these terms separately. For V_1 give by (4.1) we get

$$\begin{aligned}
V_1 &= \mathrm{Var}\int_0^n \left(f\left(\varphi_{s,t}(u)\right)-f(u)\right)N_s(du)\\
&\quad=\mathbf{E}\int_0^n\int_0^n \left(f(\varphi_{s,t}(u))-f(u)\right)\left(f\left(\varphi_{s,t}(v)\right)-f(v)\right)N_s(dv)N_s(du)\\
&\quad-\mathbf{E}\int_0^n \left(f(\varphi_{s,t}(u))-f(u)\right)N_s(du)\mathbf{E}\int_0^n \left(f\left(\varphi_{s,t}(v)\right)-f(v)\right)N_s(dv)\\
&=\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\Bigg[\int_k^{k+1}\int_l^{l+1}\mathbf{E}\left(f(\varphi_{s,t}(u))-f(u)\right)\left(f\left(\varphi_{s,t}(v)\right)-f(v)\right)\rho_s^{(2)}(u,v)dudv\\
&\quad-\int_k^{k+1}\int_l^{l+1}\mathbf{E}\left(f\left(\varphi_{s,t}(u)\right)-f(u)\right)\mathbf{E}\left(f\left(\varphi_{s,t}(v)\right)-f(v)\right)\rho_s^{(1)}(u)\rho_s^{(1)}(v)dudv\Bigg]\\
&\qquad+\sum_{k=0}^{n-1}\int_k^{k+1}\mathbf{E}\left(f\left(\varphi_{s,t}(u)\right)-f(u)\right)^2\rho_s^{(1)}(u)du.
\end{aligned}$$

Using stationarity of $\{\varphi_{s,t}(u)-u\}_{u\in\mathbb{R}}$, 1-periodicity of the function f and the property of 2-point density $\rho_s^{(2)}(u,v)=\rho_s^{(2)}(u+h,v+h)$ we can rewrite the double sum in the last expression as follows:

$$\begin{aligned}
&\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\Big[\int_k^{k+1}\int_l^{l+1}\mathbf{E}(f(\varphi_{s,t}(u))-f(u))(f(\varphi_{s,t}(v))-f(v))\rho_s^{(2)}(u,v)dudv\\
&\quad-\int_k^{k+1}\int_l^{l+1}\mathbf{E}(f(\varphi_{s,t}(u))-f(u))\mathbf{E}(f(\varphi_{s,t}(v))-f(v))\rho_s^{(1)}(u)\rho_s^{(1)}(v)dudv\Big]\\
&=\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\Big[\int_0^1\int_0^1\mathbf{E}(f(\varphi_{s,t}(u+k))-f(u+k))(f(\varphi_{s,t}(v+l))-f(v+l))\rho_s^{(2)}(u+k,v+l)dudv\\
&\quad-\int_0^1\int_0^1\mathbf{E}(f(\varphi_{s,t}(u))-f(u))\mathbf{E}(f(\varphi_{s,t}(v))-f(v))\rho_s^{(1)}(u)\rho_s^{(1)}(v)dudv\Big]\\
&=\sum_{h=-n+1}^{n-1}(n-|h|)\Big[\int_0^1\int_0^1\mathbf{E}(f(\varphi_{s,t}(u))-f(u))(f(\varphi_{s,t}(v+h))-f(v))\rho_s^{(2)}(u,v+h)dudv\\
&\quad-\int_0^1\int_0^1\mathbf{E}(f(\varphi_{s,t}(u))-f(u))\mathbf{E}(f(\varphi_{s,t}(v))-f(v))\rho_s^{(1)}(u)\rho_s^{(1)}(v)dudv\Big].
\end{aligned}$$

For fixed s,t the Brownian web $\{\varphi_{s,t}(u),u\in\mathbb{R}\}$ has the same distribution as the Arratia flow $\{x(u,t-s),\ u\in\mathbb{R}\}$. Using the upper bound [13] for α-mixing coefficient of the Arratia flow $\{x(u,t-s),\ u\in\mathbb{R}\}$ (given by Definition 2.1) [13] and the exponential decay for $\left[\rho^{(2)}(u,v+h)-\rho_s^{(1)}(u)\rho_s^{(1)}(v)\right]\le c\exp\left(-(u-v-h)^2/2s\right)$

one can get that the series

$$S := \sum_{h=0}^{\infty} h \left[\int_0^1 \int_0^1 \mathbf{E} \left(f(\varphi_{s,t}(u)) - f(u) \right) \left(f\left(\varphi_{s,t}(v+h)\right) - f(v) \right) \rho_s^{(2)}(u, v+h) du dv \right.$$
$$\left. - \int_0^1 \int_0^1 \mathbf{E} \left(f\left(\varphi_{s,t}(u)\right) - f(u) \right) \mathbf{E} \left(f\left(\varphi_{s,t}(v)\right) - f(v) \right) \rho_s^{(1)}(u) \rho_s^{(1)}(v) du dv \right]$$

converges, and there exists a constant A such that $|S| < A$. Then we get

$$\frac{1}{n} V_1 \leq \sum_{h=-n+1}^{n-1} \left[\int_0^1 \int_0^1 \mathbf{E} \left(f(\varphi_{s,t}(u)) - f(u) \right) \left(f\left(\varphi_{s,t}(v+h)\right) - f(v) \right) \rho_s^{(2)}(u, v+h) du dv \right.$$
$$\left. - \int_0^1 \int_0^1 \mathbf{E} \left(f\left(\varphi_{s,t}(u)\right) - f(u) \right) \mathbf{E} \left(f\left(\varphi_{s,t}(v)\right) - f(v) \right) \rho_s^{(1)}(u) \rho_s^{(1)}(v) du dv \right]$$
$$+ \frac{2}{n} A + \int_0^1 \mathbf{E} \left(f\left(\varphi_{s,t}(u)\right) - f(u) \right)^2 \rho_s^{(1)}(u) du.$$

Let us consider the absolute value of the sum in the last expression:

$$\left| \sum_{h=-(h-1)}^{n-1} \left[\int_0^1 \int_0^1 \mathbf{E} \left(f\left(\varphi_{s,t}(u)\right) - f(u) \right) \left(f\left(\varphi_{s,t}(v+h)\right) - f(v+h) \right) \rho_s^{(2)}(u, v+h) du dv \right.\right.$$
$$\left.\left. - \int_0^1 \int_0^1 \mathbf{E} \left(f\left(\varphi_{s,t}(u)\right) - f(u) \right) \mathbf{E} \left(f\left(\varphi_{s,t}(v)\right) - f(v) \right) \rho_s^{(1)}(u) \rho_s^{(1)}(v) du dv \right] \right|$$
$$\leq \sum_{h=-n+1}^{n-1} \int_0^1 \int_0^1 \left| \mathbf{E} \left(f\left(\varphi_{s,t}(u)\right) - f(u) \right) \left(f\left(\varphi_{s,t}(v+h)\right) - f(v+h) \right) \right|$$
$$\cdot \left| \rho_s^{(2)}(u, v+h) - \rho_s^{(1)}(u) \rho_s^{(1)}(v) \right| du dv$$
$$+ \sum_{h=-n+1}^{n-1} \int_0^1 \int_0^1 \left| \operatorname{cov} \left(f\left(\varphi_{s,t}(u)\right), f\left(\varphi_{s,t}(v+h)\right) \right) \right| \rho_s^{(1)}(u) \rho_s^{(1)}(v) du dv$$
$$\leq 2 \sum_{h=0}^{\infty} \int_0^1 \int_0^1 \sqrt{\mathbf{E} \left(f\left(\varphi_{s,t}(u)\right) - f(u) \right)^2} \sqrt{\mathbf{E} \left(f\left(\varphi_{s,t}(v+h)\right) - f(v+h) \right)^2}$$
$$\cdot \left| \rho_s^{(2)}(u, v+h) - \rho_s^{(1)}(u) \rho_s^{(1)}(v) \right| du dv$$
$$+ 2 \sum_{h=0}^{\infty} \int_0^1 \int_0^1 \left| \operatorname{cov} \left(f\left(\varphi_{s,t}(u)\right), f\left(\varphi_{s,t}(v+h)\right) \right) \right| \rho_s^{(1)}(u) \rho_s^{(1)}(v) du dv.$$

Recall, that $\rho_s^{(1)}(u) = \frac{1}{\sqrt{\pi s}}$ for every $u \in \mathbb{R}$. Now, using Lipshitz continuity of f with the Lipshitz constant L, we get the upper bound for the first summand in the

last expression:

$$L^2\sum_{h=0}^{\infty}\int_0^1\int_0^1\left(\mathbf{E}\left(\varphi_{s,t}(u)-u\right)^2\mathbf{E}\left(\varphi_{s,t}(v)-v\right)^2\right)^{1/2}\left|\rho_s^{(2)}(u,v+h)-\frac{1}{\pi s}\right|dudv$$
$$\leq (t-s)\sum_{h=0}^{\infty}\int_0^1\int_0^1\left|\rho_s^{(2)}(u,v+h)-\frac{1}{\pi s}\right|dudv\leq C|t-s|,$$

since the last series converges.

Now, let us consider $\operatorname{cov}\left(f\left(\varphi_{s,t}(u)\right),f\left(\varphi_{s,t}(v)\right)\right)$. Using boundedness of the function f by constant C, we get

$$\begin{aligned}
\left|\operatorname{cov}\left(f\left(\varphi_{s,t}(u)\right),f\left(\varphi_{s,t}(v)\right)\right)\right| &= \left|\mathbf{E}f\left(\varphi_{s,t}(u)\right)f\left(\varphi_{s,t}(v)\right)-\mathbf{E}f\left(\varphi_{s,t}(u)\right)\mathbf{E}f\left(\varphi_{s,t}(v)\right)\right|\\
&\leq \left|\mathbf{E}f\left(\varphi_{s,t}(u)\right)f\left(\varphi_{s,t}(v)\right)\mathbb{1}_{\{\varphi_{s,t}(u)=\varphi_{s,t}(v)\}}\right|\\
&+\left|\mathbf{E}f\left(\varphi_{s,t}(u)\right)f\left(\varphi_{s,t}(v)\right)\mathbb{1}_{\{\varphi_{s,t}(u)\neq\varphi_{s,t}(v)\}}-\mathbf{E}f\left(\varphi_{s,t}(u)\right)\mathbf{E}f\left(\varphi_{s,t}(v)\right)\right|\\
&\leq C^2\mathbf{P}\left(\varphi_{s,t}(u)=\varphi_{s,t}(v)\right)\\
&+\left|\mathbf{E}f\left(u+w_1(t-s)\right)f\left(v+w_2(t-s)\right)\mathbb{1}_{\{\forall r\in[0,t-s]\quad u+w_1(r)\neq v+w_2(r)\}}\right.\\
&\left.-\ \mathbf{E}f\left(u+w_1(t-s)\right)f\left(v+w_2(t-s)\right)\right|,
\end{aligned}$$

where we used that two trajectories $\{\varphi_{s,t}(u),t\geq s\}$ and $\{\varphi_{s,t}(v),t\geq s\}$ before meeting have the same distribution as trajectories of two independent Brownian motions $\{u+w_1(r),r\geq 0\},\{v+w_2(r),r\geq 0\}$ before their meeting. Now,

$$\mathbf{P}\left(\varphi_{s,t}(u)=\varphi_{s,t}(v)\right)=\mathbf{P}\left(\frac{1}{\sqrt{2}}\sup_{t\in[0,t-s]}w(r)>|u-v|\right)$$
$$=2\mathbf{P}(w(t-s)>\sqrt{2}|u-v|)\leq 2\int_{\sqrt{2}|u-v|/\sqrt{t-s}}^{\infty}e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}dx\leq\frac{\sqrt{t-s}}{|u-v|}e^{-\frac{(u-v)^2}{(t-s)}}$$

and

$$\begin{aligned}
&\left|\mathbf{E}f\left(u+w_1(t-s)\right)f\left(v+w_2(t-s)\right)\mathbb{1}_{\{\forall r\in(0,t-s):\ u+w_1(r)\neq v+w_2(r)\}}\right.\\
&\left.-\ \mathbf{E}f\left(u+w_1(t-s)\right)f\left(v+w_2(t-s)\right)\right|\\
\leq&\left|\ \mathbf{E}f\left(u+w_1(t-s)\right)f\left(v+w_2(t-s)\right)\mathbb{1}\left(\exists r\in[0,t-s]:u+w_1(r)=v+w_2(r)y\ \right|\right.\\
&\leq C^2\mathbf{P}\left(\sup_{r\in[0,t-s]}w(r)>|u-v|\sqrt{2}\right)\leq C^2\frac{\sqrt{t-s}}{|u-v|}e^{-\frac{(u-v)^2}{t-s}}
\end{aligned}$$

From this we get

$$\left|\operatorname{cov}\left(f\left(\varphi_{s,t}(u)\right),f\left(\varphi_{s,},v\right)\right)\right|\leq 2C^2\frac{\sqrt{t-s}}{|u-v|}e^{-\frac{(u-v)^2}{t-s}}$$

and

$$\sum_{h=0}^{\infty}\left|\operatorname{cov}\left(f\left(\varphi_{s,t}(u)\right),f\left(\varphi_{s,t}(v)\right)\right)\right|\leq 2C^2\sqrt{t-s}\sum_{h=0}^{\infty}\frac{1}{|u-v+h|}e^{-\frac{(u-v+h)^2}{t-s}}\leq\text{const}\sqrt{t-s}.$$

Summarizing, we get the upper bound for V_1 from (4.1):

$$\frac{1}{n}V_1 \le \sum_{h=-n+1}^{n-1}\left[\int_0^1\int_0^1 \mathbf{E}\left(f(\varphi_{s,t}(u))-f(u)\right)\left(f\left(\varphi_{s,t}(v+h)\right)-f(v)\right)\rho_s^{(2)}(u,v+h)dudv\right.$$
$$\left.-\int_0^1\int_0^1 \mathbf{E}\left(f\left(\varphi_{s,t}(u)\right)-f(u)\right)\mathbf{E}\left(f\left(\varphi_{s,t}(v)\right)-f(v)\right)\rho_s^{(1)}(u)\rho_s^{(1)}(v)dudv\right]$$
$$+\frac{2}{n}A+\int_0^1 \mathbf{E}\left(f\left(\varphi_{s,t}(u)\right)-f(u)\right)^2\rho_s^{(1)}(u)du \le c|t-s|+c\sqrt{t-s}+\frac{A}{n}.$$

Now let us consider V_2 given by (4.2):

$$V_2=\mathrm{Var}\left(\sum_{i\in\mathbb{Z}\setminus I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right)\mathbb{1}_{\{\xi_{s,i}\in[0,n]\}}\right)$$
$$\le \mathbf{E}\left(\sum_{i\in\mathbb{Z}\setminus I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right)\mathbb{1}_{\{\xi_{s,i}\in[0,n]\}}\right)^2$$
$$=\mathbf{E}\sum_{i\in\mathbb{Z}\setminus I_{s,t}}\sum_{j\in\mathbb{Z}\setminus I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right)f\left(\varphi_{s,t}\left(\xi_{s,j}\right)\right)\mathbb{1}_{\{\xi_{s,i}\in[0,n]\}}\mathbb{1}_{\{\xi_{s,j}\in[0,n]\}}$$
$$\le C^2\mathbf{E}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\mathbb{1}_{\{\exists\ l<i:\ \varphi_{s,t}(\xi_{s,i})=\varphi_{s,t}(\xi_{s,l})\}}\mathbb{1}_{\{\xi_{s,i}\in[0,n])\}}$$
$$\mathbb{1}_{\{\exists\ k<j:\ \varphi_{s,t}(\xi_{s,k})=\varphi_{s,t}(\xi_{s,j})\}}\mathbb{1}_{\{\xi_{s,j}\in[0,n])\}}$$
$$\le C^2\mathbf{E}\left(\sum_{i\in\mathbb{Z}}\sum_{l\neq i}\mathbb{1}_{\{\varphi_{s,t}(\xi_{s,i})=\varphi_{s,t}(\xi_{s,l})\}}\mathbb{1}_{\{\xi_{s,i}\in[0,n],\xi_{s,l}\in[0,n]\}}\right.$$
$$\left.+\sum_{j\in\mathbb{Z}}\sum_{k\neq j}\mathbb{1}_{\{\varphi_{s,t}(\xi_{s,k})=\varphi_{s,t}(\xi_{s,j})\}}\mathbb{1}_{\{\xi_{s,k}\in[0,n],\xi_{s,j}\in[0,n]\}}\right)$$
$$=2C^2\mathbf{E}\int_0^n\int_0^n \mathbb{1}_{\{\varphi_{s,t}(u)=\varphi_{s,t}(v)\}}N_s^{(2)}(dudv)$$
$$=2C^2\int_0^n\int_0^n \mathbf{P}\left(\varphi_{s,t}(u)=\varphi_{s,t}(v)\right)\rho_s^{(2)}\left(u,v\right)dudv$$
$$\le \text{ const }\int_0^n\int_0^n \frac{\sqrt{t-s}}{|u-v|}e^{-\frac{(u-v)^2}{(t-s)}}dudv,$$

where we used that $\rho_t^{(2)}$ is uniformly bonded by $1/(\pi t)$ (Lemma 2 in [21]). So, we get the upper bound for V_2 :

$$\frac{1}{n}V_2 \le \frac{\text{const}}{n}\sqrt{t-s}\int_0^n\int_0^n \frac{1}{|u-v|}e^{-\frac{(u-v)^2}{t-s}}dudv$$
$$=\sqrt{t-s}\frac{1}{n}\int_0^n\int_{-v}^{n-v}\frac{1}{|u|}e^{-\frac{u^2}{2}}dudv \le \sqrt{t-s}\ \text{const}.$$

Now let us consider V_4 given by (4.4):

$$V_4 = \operatorname{Var}\left(\sum_{i\in I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) \mathbb{1}_{\{\xi_{s,i}\in[0,n],\ \varphi(\xi_{s,i})\notin[0,n]\}}\right)$$
$$\leq 2\operatorname{Var}\left(\sum_{i\in\mathbb{Z}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) \mathbb{1}_{\{\xi_{s,i}\in[0,n],\ \varphi(\xi_{s,i})\notin[0,n]\}}\right)$$
$$+ 2\operatorname{Var}\left(\sum_{i\in\mathbb{Z}\setminus I_{s,t}} f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right) \mathbb{1}_{\{\xi_{s,i}\in[0,n],\ \varphi(\xi_{s,i})\notin[0,n]\}}\right)$$

The second variance in the last expression can be estimated as the variance in V_2. Now let us consider

$$\operatorname{Var}\left(\int_0^n f(u)\mathbb{1}_{\{\varphi_{s,t}(u)\notin[0,n]\}}N_s(du)\right)$$
$$= \mathbf{E}\int_0^n\int_0^n f(u)f(v)\mathbb{1}_{\{\varphi_{s,t}(u)\notin[0,n]\}}\mathbb{1}_{\{\varphi_{s,t}(v)\notin[0,n]\}}N_s(du)N_s(dv)$$
$$- \left(\mathbf{E}\int_0^n f(u)\mathbb{1}_{\{\varphi_{s,t}(u)\notin[0,n]\}}N_s(du)\right)^2$$
$$= \int_0^n\int_0^n f(u)f(v)\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n],\ \varphi_{s,t}(v)\notin[0,n]\right\}\rho_s^{(2)}(u,v)dudv$$
$$+\int_0^n f(u)f(u)\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}\rho_s^{(1)}(u)du - \left(\int_0^n f(u)\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}\rho_s^{(1)}(u)du\right)^2$$
$$= \int_0^n\int_0^n f(u)f(v)\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n],\varphi_{s,t}(v)\notin[0,n]\right\}\left(\rho_s^{(2)}(u,v)-\frac{1}{\pi s}\right)dudv$$
$$+\frac{1}{\pi s}\int_0^n\int_0^n f(u)f(v)\left[\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n],\ \varphi_{s,t}(v)\notin[0,n]\right\}\right.$$
$$\left.-\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}\right]dudv$$
$$+\int_0^n f^2(u)\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}\frac{1}{\pi s}du. \tag{4.5}$$

The first summand in the last expression we estimate as

$$\left|\int_0^n\int_0^n f(u)f(v)\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n],\varphi_{s,t}(v)\notin[0,n]\right\}\left(\rho_s^{(2)}(u,v)-\frac{1}{\pi s}\right)dudv\right|$$
$$\leq C^2\int_0^n\int_0^n\sqrt{\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}}\sqrt{\mathbf{P}\left\{\varphi_{s,t}(v)\notin[0,n]\right\}}\left|\rho_s^{(2)}(u,v)-\frac{1}{\pi s}\right|dudv.$$

Denoting a standard Wiener process as w, we get the upper bound for the proba-

bility in the last expression:

$$\begin{aligned}\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\} &= \mathbf{P}\{u+w(t-s)\notin[0,n]\}\\ &\le \mathbf{P}\{w(t-s)>n-u\}+\mathbf{P}\{w(t-s)<-u\}\\ &\le \sqrt{t-s}\left(\frac{1}{n-u}e^{-\frac{(n-u)^2}{2t-s}}\right)+\frac{1}{u}e^{-\frac{u^2}{2t-s}}\Bigg).\end{aligned}$$

Since $\left|\rho_s^{(2)}(u,v)-\frac{1}{\pi s}\right|\le e^{-\frac{(u-v)^2}{2s}}$,

$$\begin{aligned}&\int_0^n\int_0^n\sqrt{\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}\mathbf{P}\left\{\varphi_{s,t}(v)\notin[0,n]\right\}}\left|\rho_s^{(2)}(v,v)-\frac{1}{\pi s}\right|dudv\\ &\qquad\le\sqrt{t-s}\int_0^n\int_0^n\left[\frac{1}{\sqrt{n-u}}e^{-\frac{(n-u)^2}{4(t-s)}}+\frac{1}{\sqrt{u}}e^{-\frac{u^2}{4(t-s)}}\right]\\ &\qquad\quad\cdot\left[\frac{1}{\sqrt{n-v}}e^{-\frac{(n-v)^2}{4(t-s)}}+\frac{1}{\sqrt{v}}e^{-\frac{v^2}{4(t-s)}}\right]e^{-\frac{(u-v)^2}{2s}}dudv\\ &\qquad\le\sqrt{t-s}\int_0^n\int_0^n\left(\frac{1}{\sqrt{n-u}}+\frac{1}{\sqrt{u}}\right)\left(\frac{1}{\sqrt{n-v}}+\frac{1}{\sqrt{v}}\right)e^{-\frac{(u-v)^2}{2s}}dudv\end{aligned}$$

One can check that the double integral in the last expression can be estimated by $\mathrm{const}\cdot n$. The second integral in (4.5) can be estimated as:

$$\begin{aligned}&\Big|\int_0^n\int_0^n f(u)f(v)\left(\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n],\ \varphi_{s,t}(v)\notin(0,n)\right\}\right.\\ &\qquad\left.-\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}dudv\right)\Big|\\ &\le C^2\int_0^n\int_0^n\left|\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n],\ \varphi_{s,t}(v)\notin[0,n]\right\}-\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}\mathbf{P}\left\{\varphi_{s,t}(v)\notin[0,n]\right\}\right|dudv.\end{aligned}$$

Define the event $A:=\{\varphi_{s,t}(u)=\varphi_{s,t}(v)\}$. Using notation $\overline{A}$ to denote the complement of the event A we get

$$\begin{aligned}&\left|\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n],\ \varphi_{s,t}(v)\notin[0,n]\right\}-\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}\mathbf{P}\left\{\varphi_{s,t}(v)\notin[0,n]\right\}\right|\\ &\qquad\le\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n],\varphi_{s,t}(v)\notin[0,n],\ A\right\}\\ &\qquad+\Big|\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n],\varphi_{s,t}(v)\notin[0,n],\overline{A}\right\}\\ &\qquad\qquad-\mathbf{P}\left\{u+w_1(t-s)\notin[0,n],v+w_2(t-s)\notin[0,n]\right\}\Big|,\end{aligned}$$

where as before we use that two trajectories $\{\varphi_{s,t}(u),\ t\ge s\}$ and $\{\varphi_{s,t}(v),t\ge s\}$ before their meeting have the same distribution as trajectories of two independent Brownian motions $\{u+w_1(r),r\ge0\}$ and $\{v+w_2(r),r\ge0\}$ before their meeting. Continuing the estimation,

$$\begin{aligned}&\left|\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n],\ \varphi_{s,t}(v)\notin[0,n]\right\}-\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}\mathbf{P}\left\{\varphi_{s,t}(v)\notin[0,n]\right\}\right|\\ &\le\Big|\ \mathbf{P}\left\{u+w_1(t-s)\notin[0,n],\ v+w_2(t-s)\notin[0,n],\forall\ r\in[0,t-s]\quad u+w_1(r)\ne v+w_2(r)\right\}\\ &\qquad-\mathbf{P}\left(u+w_1(t-s)\notin[0,n],v+w_2(t-s)\notin(0,n)\right)\ \Big|+\mathbf{P}(A)\le2\mathbf{P}(A)\end{aligned}$$

As we have shown before,

$$\mathbf{P}\left(\varphi_{s,t}(u)=\varphi_{s,t}(v)\right)\le\frac{\sqrt{t-s}}{|u-v|}e^{-\frac{(u-v)^2}{(t-s)}}.$$

So we can get the upper bound for the second integral from (4.5):

$$\begin{aligned}C^2\int_0^n\int_0^n\big|&\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n),\ \varphi_{s,t}(v)\notin[0,n]\right\}\\&-\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}\mathbf{P}\left\{\varphi_{s,t}(v)\notin[0,n)\right\}\big|\,dudv\\&\le\text{const}\sqrt{t-s}\int_0^n\int_0^n\frac{1}{|u-v|}e^{-\frac{(u-v)}{t-s}}dudv\le\text{const}\sqrt{t-s}\cdot n,\end{aligned}$$

where the constant does not depend on n.

The third integral from (4.5) we estimate as

$$\begin{aligned}\int_0^n f^2(u)\mathbf{P}\left\{\varphi_{s,t}(u)\notin[0,n]\right\}\frac{1}{\pi s}du&\le\frac{C^2}{\pi s}\int_0^n\mathbf{P}\{u+w(t-s)\notin[0,n]\}du\\&\le\text{const}\frac{1}{\sqrt{2\pi}}\int_0^n\int_{(n-u)/\sqrt{t-s}}^\infty e^{-\frac{x^2}{2}}dxdu+\frac{1}{\sqrt{2\pi}}\int_0^n\int_{u/\sqrt{t-s}}^\infty e^{-\frac{x^2}{2}}dxdu\\&=\text{const}\frac{1}{\sqrt{2\pi}}\int_0^n\int_{u/\sqrt{t-s}}^\infty e^{-\frac{x^2}{2}}dxdv=\frac{\text{const}}{\sqrt{2\pi}}\int_0^{\sqrt[4]{t-s}}\int_{\frac{u}{\sqrt{t-s}}}^\infty e^{-\frac{x^2}{2}}dxdv\\&+\frac{\text{const}}{\sqrt{2\pi}}\int_{\sqrt[4]{t-s}}^n\int_u^\infty e^{-\frac{x^2}{2}}dxdv\le\tilde{c}\sqrt[4]{t-s}\cdot\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\int_{\sqrt[4]{t-s}}^n\frac{\sqrt{t-s}}{u}e^{-\frac{u^2}{(t-s)}}du\\&\le\text{const}\sqrt[4]{t-s}.\end{aligned}$$

Summarizing we get the upper bound for the variance V_4 :

$$\begin{aligned}\frac{1}{n}V_4=\frac{1}{n}\,\text{Var}\left(\sum_{i\in I_{s,t}}f\left(\varphi_{s,t}\left(\xi_{s,i}\right)\right)\mathbb{1}_{\{\xi_{s,i}\in[0,n],\ \varphi(\xi_{s,i})\notin[0,n]\}}\right)\\\le\text{const}\sqrt{t-s}+\text{const}\frac{1}{n}\sqrt[4]{t-s}.\end{aligned}$$

The term V_3 form (4.3) can be estimated similarly to V_4.

Summarizing obtained estimations, we get: there exist constants c and A such that for all $n\ge1$

$$\mathbf{E}\left(\frac{1}{\sqrt{n}}\left(\int_0^n f(u)N_s(du)-\int_0^n f(u)N_t(du)\right)\right)^2\le c\sqrt{t-s}+\frac{A}{n}.$$

Then for $X_t(f)$ we get

$$\mathbf{E}\left(X_t(f)-X_s(f)\right)^2\le c\sqrt{t-s},$$

where the constant c depends on a function f. Since $\{X_t(f)\}_{t\ge0}$ is a centered Gaussian process, then from this inequality, the existence of a continuous modification follows. □

Now we can formulate a limit theorem for the sequence

$$X_t^n(f) = \int_0^n f(u)N_t(du),\ t \in [t_0, T]$$

as $n \to \infty$ in the space of cadlag functions.

Theorem 4.3. *Let $f \in C(\mathbb{R})$ be 1-periodic Lipshitz continuous function such that $\int_0^1 f(u)du = 0$. Assume that $\varkappa$ is a finite signed measure on $[t_0, T]$, $t_0 > 0$ such that*

$$\int_{t_0}^T \int_{t_0}^T \left[\lim_{n\to\infty} \frac{1}{n}\mathbf{E} \int_0^n f(u)N_t(du)\ \int_0^n f(u)N_s(du)\right] \varkappa(dt)\varkappa(ds) > 0.$$

Then

$$\int_{t_0}^T X_t^n(f)\varkappa(dt) \Rightarrow \int_{t_0}^T X_t(f)\varkappa(dt),\ n \to \infty,$$

where $X_t(f)$ is a Gaussian process with zero mean and covariance function from Theorem 2.1:

$$c_{s,t} = \lim_{n\to\infty} \frac{1}{n}\mathrm{cov}\ (X_s^n(f), X_t^n(f))\,.$$

Proof. Changing the order of integration we get:

$$\int_{t_0}^T X_t^n(f)\varkappa(dt) = \frac{1}{\sqrt{n}} \int_{t_0}^T \int_0^n f(u)N_t(du)\varkappa(dt) = \frac{1}{\sqrt{n}} \int_0^n f(u) \int_{t_0}^T N_t(du)\varkappa(dt)$$
$$= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \int_k^{k+1} f(u) \int_{t_0}^T N_t(du)\varkappa(dt).$$

Let us consider the sequence

$$\left\{ Y_k = \int_k^{k+1} f(u) \int_{t_0}^T N_t(du)\varkappa(dt) \right\}_{k\geq 0}.$$

By corollary 4.1 the measure $\nu(\cdot) = \int_{t_0}^T N_t(\cdot)\varkappa(dt)$ is well defined. From the representation

$$\int_{t_0}^T N_t(f)\varkappa(dt) = \lim_{n\to\infty} \sum_{k=-n2^n}^{n2^n} \int_{t_0}^{\left(T\wedge\sigma_k^{(n)}\right)\vee t_0} f\left(x\left(\frac{k}{2^n}, t\right)\right)\varkappa(dt)$$

we get stationarity of the random measure ν. Namely, since the process $\{x(u,t) - u,\ t \in [0,\infty)\}$ is stationary in $C([0,\infty))$, then for $\Delta_1, \ldots, \Delta_m$ the random vector $(\nu(\Delta_1)\ldots, \nu(\Delta_m))$ is equal by distribution to $(\nu(\Delta_1 + h)\ldots, \nu(\Delta_m + h))$. From this follows that the sequence $\{Y_k\}_{k\geq 0}$ is stationary. The expectation of the measure ν is

$$\mathbf{E}\nu(\Delta) = \mathbf{E}\int_{t_0}^T N_t(\Delta)\varkappa(dt) = \int_{t_0}^T \mathbf{E}N_t(\Delta)\varkappa(dt)$$
$$= \int_{t_0}^T \int_\Delta \rho_t^{(1)}(u)du\varkappa(dt) = \int_\Delta \int_{t_0}^T \frac{1}{\sqrt{\pi t}}\varkappa(dt)du$$

So, the measure $\mathbf{E}\nu$ has a density equal to the constant $\rho^{(1)}_{\varkappa}(u) = \int_{t_0}^{T} \frac{1}{\sqrt{\pi t}} \varkappa(dt)$. Next, note that

$$|Y_k| = \left| \int_k^{k+1} f(u) \int_{t_0}^{T} N_t(du)\varkappa(dt) \right| \le \max_{[0,1]} |f| \max_{[t_0,T]} N_t([k,k+1])|\varkappa|([t_0,T]),$$

where $|\varkappa|$ is the total variation of the measure $\varkappa$. So we can write

$$\mathbf{E}Y_k = \mathbf{E} \int_k^{k+1} f(u) \int_{t_0}^{T} N_t(du)\varkappa(dt) = \int_{t_0}^{T} \mathbf{E} \left[\int_k^{k+1} f(u) N_t(du) \right] \varkappa(dt) = 0.$$

To find second moment $\mathbf{E}Y_1^2$, consider

$$\mathbf{E} \left[\int_{t_0}^{T} \int_0^1 f(u) N_t(du)\varkappa(dt) \right]^2 = \int_{t_0}^{T} \int_{t_0}^{T} \mathbf{E} \left[\int_0^1 f(u) N_t(du) \int_0^1 f(u) N_s(du) \right] \varkappa(dt)\varkappa(ds)$$

Using Lemma 3.3 we get the expression for $\mathbf{E}\left[\int_0^1 f(u)N_t(du) \int_0^1 f(u)N_s(du)\right]$ with $s < t$:

$$\begin{aligned}
&\mathbf{E} \left[\int_0^1 f(u) N_t(du) \int_0^1 f(u) N_s(du) \right] \\
&\qquad = \mathbf{E} \left[\int_0^1 f(u) N_s(du) \mathbf{E} \left(\int_0^1 f(u) N_t(du) | \mathcal{F}_s^x \right) \right] \\
&= \mathbf{E} \int_0^1 f(u) N_s(du) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \int_{\mathbb{R}^k} \left(\int_0^1 f(v) q_{t-s}(u_*, u^*, v) dv \right) N_s^{(k)}(d\vec{u}) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \mathbf{E} \left[\int_0^1 f(u) N_s(du) \int_{\mathbb{R}^k} \left(\int_0^1 f(v) q_{t-s}(u_*, u^*, v) dv \right) N_s^{(k)}(d\vec{u}) \right] \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \Big[\int_0^1 \int_{\mathbb{R}^k} f(y) \int_0^1 f(v) q_{t-s}(u_*, u^*, v) dv \rho_s^{(k+1)}(y, u_1, \ldots, u_k) d\vec{u} dy \\
&\qquad + k \int_0^1 \int_{\mathbb{R}^{k-1}} f(u_1) \int_0^1 f(v) q_{t-s}(u_*, u^*, v) dv \rho_s^{(k)}(u_1, \ldots, u_k) d\vec{u} \Big],
\end{aligned}$$

where $\rho_s^{(k)}$ is the $k-$point density of the point process $N_s(\cdot)$.

The same argument as in Lemma 2.1 and Remark 2.1 gives us the $\alpha-$mixing coefficient of the sequence $\{Y_k\}_{k\ge 0}$ satisfies the inequality (with the constant c which differs from the constant in Lemma 2.1)

$$\alpha_Y(h) \le \frac{c\sqrt{T}}{h} \exp(-\frac{h^2}{8T}).$$

In order to apply the central limit theorem for a sequence of weakly dependent random variables [16], we should guarantee that $\lim_{n\to\infty} \frac{1}{n} \mathrm{Var} \sum_{k=1}^{n} Y_k > 0$. Let us

consider

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Var}\sum_{k=1}^{n}Y_k=\lim_{n\to\infty}\frac{1}{n}\mathbf{E}\left(\int_{t_0}^{T}\int_0^n f(u)N_t(du)\varkappa(dt)\right)^2$$
$$=\lim_{n\to\infty}\frac{1}{n}\int_{t_0}^{T}\int_{t_0}^{T}\left[\mathbf{E}\int_0^n f(u)N_t(du)\ \int_0^n f(u)N_s(du)\right]\varkappa(dt)\varkappa(ds). \quad (4.6)$$

In order to change the limit and integral using dominated convergence theorem, note that

$$\left|\frac{1}{n}\mathbf{E}\int_0^n f(u)N_t(du)\ \int_0^n f(u)N_s(du)\right|\le\frac{1}{n}\mathbf{E}\left|\int_0^n f(u)N_t(du)\ \int_0^n f(u)N_s(du)\right|$$
$$\le\frac{1}{n}\left(\mathbf{E}\left(\int_0^n f(u)N_t(du)\right)^2+\mathbf{E}\left(\int_0^n f(u)N_s(du)\right)^2\right)$$
$$=\frac{1}{n}\left(\mathrm{Var}\int_0^n f(u)N_t(du)+\mathrm{Var}\int_0^n f(u)N_s(du)\right).$$

Recall, that by the formula (1.4)

$$\mathrm{Var}\int_0^n f(u)N_t(du)=\int_0^n\int_0^n f(v_1)f(v_2)\Big(\rho_t^{(2)}(v_1,v_2)-\frac{1}{\pi t}\Big)dv_1\,dv_2+\frac{1}{\sqrt{\pi t}}\int_0^n f^2(v_1)dv_1$$

and using (1.3),

$$\rho_t^{(2)}(v_1,v_2)-\frac{1}{\pi t}$$
$$=\frac{1}{\pi t}\left(\frac{|v_2-v_1|}{2\sqrt{t}}\cdot e^{-(v_2-v_1)^2/4t}\cdot\int\limits_{|v_2-v_1|/\sqrt{t}}^{+\infty}e^{-v^2/4}\,dv-e^{-(v_2-v_1)^2/2t}\right).$$

From this, for every $n\ge 1$,

$$\frac{1}{n}\int_0^n\int_0^n\left(\rho_t^{(2)}(v_1,v_2)-\frac{1}{\pi t}\right)dv_1dv_2\le c\frac{1}{\sqrt{t}}$$

Now, applying the dominated convergence theorem in (4.6), we get

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Var}\sum_{k=1}^{n}Y_k$$
$$=\int_{t_0}^{T}\int_{t_0}^{T}\left[\lim_{n\to\infty}\frac{1}{n}\mathbf{E}\int_0^n f(u)N_t(du)\ \int_0^n f(u)N_s(du)\right]\varkappa(dt)\varkappa(ds)>0, \quad (4.7)$$

by the assumption. By the central limit theorem for a weakly dependent stationary random sequence [16], it follows that

$$\int_{t_0}^{T}X_t^n(f)\varkappa(dt)\Rightarrow\xi_\varkappa,\ n\to\infty.$$

where $\xi_\varkappa$ is a centered Gaussian random variable with variance given by (4.7). In order to prove that the limit $\xi_\varkappa$ can be represented as $\int_{t_0}^T X_t(f)\varkappa(dt)$, where $X_t(f)$ is a Gaussian process we consider an approximation of the measure $\varkappa$. Define the sequence of measures $\{\varkappa_m\}_{m\geq 1}$ by

$$\varkappa_m = \sum_{j=0}^{m-2} \varkappa([t_j^{(m)}, t_{j+1}^{(m)}))\delta_{t_j^{(m)}} + \varkappa([t_{m-1}^{(m)}, t_m^{(m)}])\delta_{t_m^{(m)}},$$

where $\Pi_m = \{t_0 = t_0^{(m)} \leq \ldots \leq t_m^{(m)} = T\}$ is a sequence of partitions of the interval $[t_0, T]$ with $\|\Pi_m\| = \max_{j=0,\ldots,m-1} |t_{j+1}^{(m)} - t_j^{(m)}| \to 0$ as $m \to \infty$. By Theorem 2.1 (2) for the measures $\varkappa_m$, $(m \geq 1)$ we have

$$\int_{t_0}^T X_t^n(f)\varkappa_m(dt) = \sum_{k=1}^M a_k X^n_{t_k^{(m)}}(f) \Rightarrow \sum_{k=1}^M a_k X_{t_k^{(m)}}(f),\ n \to \infty,$$

where $\{X_{t_k^{(m)}}\}_{k=1}^M$ are centered Gaussian random variables with covariance function

$$c_{s,t} = \lim_{n\to\infty} \frac{1}{n} \mathrm{cov}\,(X_s^n(f), X_t^n(f))\,.$$

By Theorem 4.2, the Gaussian process with this covariance function has a continuous modification, so the integral $\int_{t_0}^T X_t(f)\varkappa_m(dt)$ is well defined, and we can write

$$\int_{t_0}^T X_t^n(f)\varkappa_m(dt) \Rightarrow \int_{t_0}^T X_t(f)\varkappa_m(dt),\ n \to \infty.$$

From the proof of Theorem 4.2 we get that process $X_t(f)$ satisfies

$$\mathbf{E}(X_t(f) - X_s(f))^2 \leq C\sqrt{t-s}.$$

Using this we can estimate:

$$\begin{aligned}
\mathbf{E}\left|\int_{t_0}^T X_t(f)(\varkappa - \varkappa_m)(dt)\right| &\leq \mathbf{E}\sum_{j=0}^{m-1}\int_{t_j^{(m)}}^{t_{j+1}^{(m)}} \left|X_t(f) - X_{t_j^{(m)}}(f)\right| |\varkappa|(dt) \\
&= \sum_{j=0}^{m-1}\int_{t_j^{(m)}}^{t_{j+1}^{(m)}} \mathbf{E}\left|X_t(f) - X_{t_j^{(m)}}(f)\right| |\varkappa|(dt) \\
&\leq \sum_{j=0}^{m-1} \sqrt{\varkappa([t_j^{(m)}, t_{j+1}^{(m)}))}\sqrt{\int_{t_j^{(m)}}^{t_{j+1}^{(m)}} \left[\mathbf{E}\left|X_t(f) - X_{t_j^{(m)}}(f)\right|\right]^2 |\varkappa|(dt)} \\
&\leq \sum_{j=0}^{m-1} \sqrt{\varkappa([t_j^{(m)}, t_{j+1}^{(m)}))}\sqrt{\int_{t_j^{(m)}}^{t_{j+1}^{(m)}} \mathbf{E}\left|X_t(f) - X_{t_j^{(m)}}(f)\right|^2 |\varkappa|(dt)}
\end{aligned}$$

$$\leq \sum_{j=0}^{m-1} \sqrt{\varkappa([t_j^{(m)}, t_{j+1}^{(m)}))} \sqrt{\int_{t_j^{(m)}}^{t_{j+1}^{(m)}} \sqrt{t - t_j^{(m)}} |\varkappa|(dt)}$$

$$\leq C \sum_{j=0}^{m-1} |\varkappa|([t_j^{(m)}, t_{j+1}^{(m)})) \sqrt[4]{t_{j+1}^{(m)} - t_j^{(m)}} \to 0,\ m \to \infty.$$

For any $\varphi \in C_b(\mathbb{R}) \cap Lip(\mathbb{R})$, and any ε we can find such $m,\ n$ that

$$\left| \mathbf{E}\varphi\left(\int_{t_0}^T X_t(f)\varkappa(dt)\right) - \mathbf{E}\varphi\left(\int_{t_0}^T X_t^n(f)\varkappa(dt)\right)\right|$$

$$\leq \left|\mathbf{E}\varphi\left(\int_{t_0}^T X_t(f)\varkappa(dt)\right) - \mathbf{E}\varphi\left(\int_{t_0}^T X_t(f)\varkappa_m(dt)\right)\right|$$

$$+ \left|\mathbf{E}\varphi\left(\int_{t_0}^T X_t(f)\varkappa_m(dt)\right) - \mathbf{E}\varphi\left(\int_{t_0}^T X_t^n(f)\varkappa_m(dt)\right)\right|$$

$$+ \left|\mathbf{E}\varphi\left(\int_{t_0}^T X_t^n(f)\varkappa_m(dt)\right) - \mathbf{E}\varphi\left(\int_{t_0}^T X_t^n(f)\varkappa(dt)\right)\right|$$

$$\leq L\mathbf{E}\left|\int_{t_0}^T X_t(f)(\varkappa - \varkappa_m)(dt)\right|$$

$$+ \left|\mathbf{E}\varphi\left(\int_{t_0}^T X_t(f)\varkappa_m(dt)\right) - \mathbf{E}\varphi\left(\int_{t_0}^T X_t^n(f)\varkappa_m(dt)\right)\right|$$

$$+ L\mathbf{E}\left|\int_{t_0}^T X_t^n(f)(\varkappa - \varkappa_m)(dt)\right| \leq c\varepsilon$$

Consequently,

$$\int_{t_0}^T X_t^n(f)\varkappa(dt) \Rightarrow \int_{t_0}^T X_t(f)\varkappa(dt),\ n \to \infty. \qquad \square$$

5. Limiting Gaussian process as a functional.

Statement 3 from Theorem 2.1 gives us the possibility to define finite-dimensional distributions of the process ζ as a process indexed by the functions $\varphi \in L_3([0,1])$. Namely it was proved that for functions $\varphi_1, ..., \varphi_m \in L_3([0,1])$ the weak convergence of the random vector $(X_t^n(\varphi_1), ..., X_t^n(\varphi_m))$ to the random vector $(\zeta_{\varphi_1}(t), \ldots, \zeta_{\varphi_m}(t))$ holds. By the Kolmogorov theorem, one can define the Gaussian random field ζ on the space of parameters $L_3([0,1])$. Note that the covariance of ζ is

$$\text{cov }(\zeta_f(t), \zeta_g(t)) = \frac{1}{2}\int_0^1 \int_0^1 f(u)g(v)(G_t(u,v) + G_t(v,u))dudv$$

$$+ \frac{1}{\sqrt{2\pi}}\int_0^1 f(u)g(u)du = (f, (1 + \tilde{G}_t)g),$$

where we denoted by $\tilde{G}_t$ the integral operator in L_2 with kernel $\tilde{G}_t = \frac{1}{2}(G_t(u,v) + G_t(v,u))$.

Lemma 5.1. *The function $\tilde{G}_t$ is continuous.*

Proof. From Theorem 2.1, the function G_t is equal to the series

$$G_t(v_1,v_2) = g_t(v_1-v_2) + 2\sum_{l=1}^{\infty} g_t(v_1-v_2+l), \text{ where } g_t(v_1-v_2) = \rho_t^{(2)}(v_1,v_2) - \frac{1}{\pi t}.$$

Using formula (2.1) for g_t by the Weierstrass criterion for uniform convergence with upper bound
$\frac{K+l}{2\sqrt{t}} \cdot e^{-(l-K)^2/4t} \cdot \int\limits_{|K+l|/\sqrt{t}}^{+\infty} e^{-v^2/4}\,dv + e^{-(L-K)^2/2t}$, the series $\sum_{l=1}^{\infty} g_t(x+l)$ converges uniformly on the interval $[-K,K]$. □

Since for every fixed $t \geq 0$, the function $\tilde{G}_t$ is continuous, ζ can be uniquely expanded to a Gaussian random field defined on the set of the functions $L_2([0,1])$.

We will consider ζ as a generalized random element in Hilbert space $L_2([0,1])$ ([22, 5]) with the covariance operator $1 + \tilde{G}_t$. One of the most known examples of a generalized Gaussian random element on L_2 is a formal derivative dw of the 1-dimensional Wiener process $w(\cdot)$, which is often called white noise. Note that for this formal derivative, one can consider stochastic integral and multiple stochastic integrals for function from $L_2([0,1]^k)$. The definition of the integrals

$$\int_0^1 \int_0^{t_1} \dots \int_0^{t_{k-1}} f(t_1,t_2,\dots,t_k) dw(t_1)\dots dw(t_k)$$

is obvious. In the context of our paper, one can expect that a sequence multiple integrals with respect to point measure N_t after suitable normalization converges to some integrals defined by ζ. First of all, let us describe the construction of multiple integrals with respect to ζ. In the articles by A. V. Skorokhod [22] and by A.A. Dorogovtsev [5], the authors proposed to describe such integrals in terms of the Hilbert-Schmidt forms from generalized Gaussian elements. Let us represent the suitable construction.

Now we will discuss the possibility of defining the Hilbert-Schmidt form of the process ζ. Let H be a real separable Hilbert space. Consider generalized Gaussian random element ξ in H with zero mean and covariance operator A. We denote by $H_{symm}^{\otimes k}$ the space of symmetric $k-$linear Hilbert-Schmidt forms of H. The Wick product (see, for example, [5]) of Gaussian random variables we denote by $*$.

Lemma 5.2. *Assume that the covariance operator A of a generalized Gaussian element ξ is invertible. Let $A_k \in H_{symm}^{\otimes k}$. Then for every orthonormal basis $\{e_j\}_{j\geq 1}$ in H the series*

$$A_k(\xi,\dots,\xi) := \sum_{n_1,\dots,n_k=1}^{\infty} A(e_{n_1},\dots,e_{n_k})\xi(e_{n_1}) * \dots * \xi(e_{n_k}))$$

converges in L_2 and its value $A_k(\xi,\dots,\xi)$ does not depend on choice of a basis in H.

Proof. Let A_k be a finite-dimensional form that is there exists a basis $\{e_j\}_{j\geq 1}$ in H such that $A_k=\sum_{n_1,\dots,n_k=1}^{N} a_{n_1,\dots,n_k}e_{n_1}\otimes\dots\otimes e_{n_k}$. Let us define $\xi_0=A^{-1/2}\xi$ by the rule

$$\xi_0(e)=\xi(A^{-1/2}e),\ e\in H.$$

Then for finite-dimensional symmetric form A_k

$$A_k(\xi,\dots,\xi)=A_k(A^{1/2}\xi_0,\dots A^{1/2}\xi_0)$$

and

$$\mathbf{E}A_k(\xi,\dots,\xi)^2=k!\|A_k(A^{1/2}\cdot,\dots A^{1/2}\cdot)\|_k^2,$$

where $\|\cdot\|_k$ is the Hilbert-Schmidt norm. Note that $\|A_k(A^{1/2}\cdot,\dots A^{1/2}\cdot)\|_k\leq\|A^{1/2}\|^k\|A_k\|_k$. From this upper bound and the fact that the definition of $A_k(A^{1/2}\cdot,\dots A^{1/2}\cdot)$ does not depend on the choice of a basis, the statement of the lemma follows. □

From Lemma 5.1 it follows that the integral operator with kernel $\tilde{G}$ is a Hilbert-Schmidt operator, so it has a discrete spectrum with eigenvalues of finite multiplicity. From this follows $\dim Ker(1+\tilde{G})<+\infty$. Using Lemma 5.2, we can define the action of $k-$linear Hilbert-Schmidt form on limiting Gaussian element ζ since its covariance operator $1+\tilde{G}$ is invertible on $Ker^{\perp}(1+\tilde{G})$.

The next theorem gives us the limit of multiple integrals. Before we formulate it, we need to define a $k-$linear form associated with a function f. Let $f:\mathbb{R}^k\to\mathbb{R}$ be a symmetric function from $L_2([0,1]^k)$. Then, for an orthogonal basis $\{e_j\}_{j\geq 1}$ in $L_2([0,1])$ it can be written as the series

$$f=\sum_{i_1,\dots,i_k} a_{i_1,\dots,i_k}e_{i_1}\otimes\dots\otimes e_{i_k}.$$

We denote by A_f the $k-$linear Hilbert-Schmidt form on $L_2([0,1])$ which is represented with the same series

$$A_f=\sum_{i_1,\dots,i_k} a_{i_1,\dots,i_k}e_{i_1}\otimes\dots\otimes e_{i_k},$$

that is $A_f(h_1,\dots,h_k)=\sum_{i_1,\dots,i_k}a_{i_1,\dots,i_k}(e_{i_1},h_1)\dots(e_{i_k},h_k)$. As was mentioned above, the action of the form A_f on the generalized Gaussian element ζ is well defined. Also, we denote by

$$A_f(\zeta,\dots,\zeta,\delta_{x_j},\dots,\delta_{x_k})=\sum_{i_1,\dots,i_k}a_{i_1,\dots,i_k}\zeta(e_{i_1})*\dots*\zeta(e_{i_{j-1}})e_{i_j}(x_j)\dots e_{i_k}(x_k).$$

In the term of multi-linear forms from the Gaussian element ζ, we can describe the limit distribution of the multiple integral with respect to the point measure of the Arratia flow N_t.

Theorem 5.1. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a symmetric 1-periodic function such that $f|_{[0,1]^k} \in L_2([0,1]^k)$ and $\int_0^1 f(\vec{x})dx_j = 0$, $j = 1, \dots, k$. Then for $k \in 2\mathbb{Z}$*

$$\frac{1}{n^{k/2}} \int_0^n \overset{k}{\dots} \int_0^n f(\vec{x}) N_t^{(k)}(dx_1 \dots dx_k) \Rightarrow A_f(\zeta, \dots, \zeta)$$
$$+ \int_0^1 \int_0^1 C_k^2 A_f(\zeta, \dots, \zeta, \delta_{x_{k-1}}, \delta_{x_k}) G_t(x_{k-1}, x_k) dx_k dx_{k-1}$$
$$+ \dots + \int_0^1 \dots \int_0^1 \frac{k!}{(k/2)! 2^{k/2}} A_f(\delta_{x_1}, \dots \delta_{x_k}) G_t(x_1, x_2) \dots G_t(x_{k-1}, x_k) d\vec{x}, \ n \to \infty$$

and for $k \in \mathbb{Z} \setminus 2\mathbb{Z}$

$$\frac{1}{n^{k/2}} \int_0^n \overset{k}{\dots} \int_0^n f(\vec{x}) N_t^{(k)}(dx_1 \dots dx_k) \Rightarrow A_f(\zeta, \dots, \zeta)$$
$$+ \int_0^1 \int_0^1 C_k^2 A_f(\zeta, \dots, \zeta, \delta_{x_{k-1}}, \delta_{x_k}) G_t(x_{k-1}, x_k) dx_k dx_{k-1} \dots$$
$$+ \dots + \int_0^1 \dots \int_0^1 \frac{k!}{((k-1)/2)! 2^{(k-1)/2}} A_f(\zeta, \delta_{x_2}, \dots \delta_{x_k}) G_t(x_2, x_3) \dots G_t(x_{k-1}, x_k) d\vec{x},$$
$$n \to \infty.$$

Proof. We prove the statement using mathematical induction with respect to dimension k. Consider $k = 2$. Let $\{e_j\}_{j \geq 0}$ be a orthonormal basis in $L_2([0,1])$ such that $\int_0^1 e_i(x)dx = 0$ for $i \geq 1$ and $e_0 = 1$. Then function can be represented as $f|_{[0,1]^2} = \sum_{i,j=1}^{\infty} c_{ij} e_i \otimes e_j$. Denote by φ_j the periodic extension on $\mathbb{R}$ of the function e_j. Then we have

$$\int_0^n \int_0^n f(x,y) N_t^{(2)}(dxdy) = \int_0^n \int_0^n f(x,y) N_t(dx) N_t(dy) - \int_0^n f(x,x) N_t(dx)$$
$$= \sum_{i,j}^{\infty} c_{ij} \int_0^n \varphi_i(x) N_t(dx) \int_0^n \varphi_j(y) N_t(dy) - \int_0^n f(x,x) N_t(dx).$$

Using Theorem 2.1 we have

$$\frac{1}{n} \int_0^n \varphi_i(x) N_t(dx) \int_0^n \varphi_j(y) N_t(dy) \Rightarrow \zeta(e_i)\zeta(e_j), \ n \to \infty$$

and $\frac{1}{n} \int_0^n f(x,x) N_t(dx) \Rightarrow \frac{1}{\sqrt{\pi t}} \int_0^1 f(x,x)dx$, $n \to \infty$. From this follows

$$\frac{1}{n} \int_0^n \int_0^n f(x,y) N_t^{(2)}(dxdy)$$
$$= \sum_{i,j}^{\infty} c_{ij} \frac{1}{n} \int_0^n \varphi_i(x) N_t(dx) \int_0^n \varphi_j(y) N_t(dy) - \frac{1}{n} \int_0^n f(x,x) N_t(dx)$$
$$\Rightarrow \sum_{i,j}^{\infty} c_{ij} \zeta(e_i)\zeta(e_j) - \frac{1}{\sqrt{\pi t}} \int_0^1 f(x,x)dx, \ n \to \infty.$$

By the definition of Wick product, the sum in the last expression is written as

$$\sum_{i,j}^{\infty} c_{ij}\zeta(e_i)\zeta(e_j) - \frac{1}{\sqrt{\pi t}}\int_0^1 f(x,x)dx$$
$$= \sum_{i,j}^{\infty} c_{ij}(\zeta(e_i) * \zeta(e_j) + \frac{1}{2}\int_0^1\int_0^1 e_i(x)e_j(y)(G_t(x,y)+G_t(y,x))dxdy$$
$$= A_f(\zeta,\zeta) + \int_0^1\int_0^1 f(x,y)G_t(x,y)dxdy,$$

where in the last equality, we used symmetry of function f.

Now assume that we know the statement of the theorem for $k-1$. To do the induction step, we rewrite the $k-$multiple integral with respect to $N^{(k)}$ in terms of $(k-1)-$multiple integral. In the following, we can take a basis of L_2 with step functions e_j and write

$$\int_0^1\overset{k}{\cdots}\int_0^1 e_1(x_1)\dots e_k(x_k)N^{(k)}(dx_1\dots dx_k)$$
$$= \sum_{i_1}\sum_{i_2\neq i_1}\cdots\sum_{i_k\neq i_{k-1},\dots,i_1} e_1(u_{i_1})\dots e_k(u_{i_k})$$
$$= \sum_{i_1}\sum_{i_2\neq i_1}\cdots\sum_{i_{k-1}\neq i_{k-2},\dots,i_1}\left(\sum_{i_k} e_1(u_{i_1})\dots e_k(u_{i_k}) - \sum_{j=1}^{k-1} e_1(u_{i_1})\dots e_k(u_{i_j})\right)$$
$$= \int_0^1\overset{k-1}{\cdots}\int_0^1 e_1(x_1)\dots e_{k-1}(x_{k-1})N^{(k-1)}(dx_1\dots dx_{k-1})\int_0^1 e_k(x)N(dx)$$
$$- \sum_{j=1}^{k-1}\int_0^1\overset{k-1}{\cdots}\int_0^1 e_1(x_1)\dots e_{k-1}(x_{k-1})e_k(x_j)N^{(k-1)}(dx_1\dots dx_{k-1}).$$

Using the limit behavior for $(k-1)-$multiple integral, we get the statement of the theorem. □

Corollary 5.1. *For the function f satisfying conditions of the previous theorem and off-diagonal (i.e. $f(\vec{x}) = 0$ if $x_i = x_j$ for some $i \neq j$) we have*

$$\frac{1}{n^{k/2}}\int_0^n\overset{k}{\cdots}\int_0^n f(\vec{x})N^{(k)}(dx_1\dots dx_k) \Rightarrow A_f(\zeta,\dots,\zeta),\ n\to\infty.$$

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