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ON THE APPROXIMATIONS OF POINT MEASURES ASSOCIATED WITH THE BROWNIAN WEB BY MEANS OF THE FRACTIONAL STEP METHOD AND DISCRETIZATION OF THE INITIAL INTERVAL

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We establish the rate of weak convergence in the fractional step method for the Arratia flow in terms of the Wasserstein distance between the images of the Lebesgue measure under the action of the flow. We introduce finite-dimensional densities that describe sequences of collisions in the Arratia flow and derive an explicit expression for them. For the discretized initial interval, we also discuss the convergence of the corresponding approximations of the point measure associated with the Arratia flow in terms of these densities.

1. Introduction

In the present article, we consider point measures constructed from the Arratia flow and its approximations [2, 3, 8, 9]. Two types of discrete measures can be associated with a stochastic flow $\{X(u,t) \mid t \geq 0,\ u \in \mathbb{R}\}$ with coalescence on the real line: the first measure is the image of the Lebesgue measure under the action of the flow

$$\mu_t = \lambda \circ (X(\cdot, t))^{-1},$$

and the second measure is the counting measure defined by the rule

$$\nu_t(\Delta) = \big|X(\mathbb{R}, t) \cap \Delta\big|, \quad \Delta \in \mathcal{B}(\mathbb{R}).$$

Both measures are supported by the same locally finite countable set. The structure of these random measures was studied in [5–7, 14, 15]. In the first part of the article, we consider the Arratia flow with drift. This flow consists of coalescing Brownian motions with diffusion 1 and drift a, where a is a bounded Lipschitz continuous function. A stochastic flow of this kind was obtained in [2] by applying the fractional step method [1, 10] to the Brownian web [8, 9] and an ordinary differential equation driven by a. In the present work, the study of this approximation scheme is continued by discussing the rate of convergence of the images of the Lebesgue measure.

We start with recalling the fractional step method for the Brownian web proposed in [2]. Let a be a bounded Lipschitz continuous function on the real line. We consider a sequence of partitions $\big\{0 = t_0^{(n)} < \ldots < t_n^{(n)} = 1\big\}$ of the interval $[0; 1]$ with a mesh size δ_n convergent to 0 and define a family of transformations of $\mathbb{R}$

$$d\mathcal{A}_{s,t}(u) = a\big(\mathcal{A}_{s,t}(u)\big)dt,$$

$$\mathcal{A}_{s,s}(u) = u, \quad t \geq s.$$

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Given a Brownian web $\{\Phi_{s,t}(u) \mid 0 \le s \le t,\ u \in \mathbb{R}\}$ [8, 9], one can consider $\{\Phi_{s,t}\}_{0\le s\le t}$ as random mappings of $\mathbb{R}$ into itself. We set $\Delta_j^{(n)} = \big[t_j^{(n)}; t_{j+1}^{(n)}\big),\ j = \overline{0, n-1}$, and define, for $u \in \mathbb{R},\ t \in \Delta_j^{(n)}$,

$$\Phi_t^{(n)}(u) = \Phi_{t_j^{(n)},t} \circ \mathcal{A}_{t_j^{(n)},t_{j+1}^{(n)}}\Big(\mathop{\circ}_{l=0}^{j-1}(\Phi_{t_l^{(n)},t_{l+1}^{(n)}} \circ \mathcal{A}_{t_l^{(n)},t_{l+1}^{(n)}})(u)\Big),$$

$$\Phi_1^{(n)}(u) = \lim_{t\to 1-} \Phi_t^{(n)}(u).$$

The sign $\circ$ stands for the composition of functions: $f \circ g = f(g)$. The main result of [2] states that, given $u_1, \dots, u_m \in \mathbb{R}$, we have

$$\big(\Phi^{(n)}(u_1), \dots, \Phi^{(n)}(u_m)\big) \underset{n\to\infty}{\Rightarrow} \big(\Phi^a(u_1), \dots, \Phi^a(u_m)\big) \tag{1.1}$$

in the Skorokhod space $\big(D([0;1])\big)^m$, where $\{\Phi_s^a(u) \mid s \ge 0,\ u \in \mathbb{R}\}$ is an Arratia flow with drift a [3] (§ 7.3). It was proved in [2] (Proposition 1.5) that the sequence on the left-hand side of (1.1) converges only weakly, in contrast to the application of the fractional step method to ordinary SDEs [1, 10].

Let λ be the Lebesgue measure on $[0;1]$. We can define the images of λ under the mappings $\Phi_t^a,\ \Phi_t^{(n)}$ as follows:

$$\mu_t = \lambda \circ \big(\Phi_t^a\big)^{-1}, \qquad \mu_t^{(n)} = \lambda \circ \big(\Phi_t^{(n)}\big)^{-1}, \quad n \in \mathbb{N}.$$

These random measures, along with associated point processes, are central objects of the present paper in the first part of which we establish an estimate for the rate of convergence of the laws of $\big\{\mu_t^{(n)}\big\}_{n\ge 1}$ to the law of μ_t for fixed t in terms of an appropriate Wasserstein distance.

Our approach is based on the ideas from [4]. We now recall the definition of the Wasserstein distance between two probability measures. Let X be a separable complete metric space with the metric d and the corresponding Borel σ-field. The set $\mathcal{M}_p(X)$ of all probability measures μ on X such that

$$\int_X d(u,v)^p \mu\,(dv) < +\infty$$

for some (and, therefore, for any) point u, is a separable metric space [16] (Theorem 6.18) w.r.t. the distance

$$W_p(\mu_1,\mu_2) = \left(\inf_{\varkappa\in\Pi(\mu_1,\,\mu_2)} \int_{X^2} d(u,v)^p \varkappa(du,dv)\right)^{1/p}, \quad p \ge 1,$$

where $\Pi(\mu_1,\mu_2)$ is the set of all probability measures on X^2 with marginals μ_1 and μ_2.

The measures $\mu_t,\ \mu_t^{(n)},\ n \in \mathbb{N}$, are random elements in $\mathcal{M}_p(\mathbb{R})$ for any $p \ge 1$. Let L_t and $L_t^{(n)}$ be the laws of μ_t and $\mu_t^{(n)}$ in $\mathcal{M}_1\big(\mathcal{M}_p(\mathbb{R})\big)$, respectively. For fixed p, the corresponding Wasserstein distance between the probability measures $L', L'' \in \mathcal{M}_1\big(\mathcal{M}_p(\mathbb{R})\big)$ is defined as

$$W_1(L', L'') = \inf \mathrm{E}\, W_p(\mu', \mu''),$$

where the infimum is taken over the set of pairs of $\mathcal{M}_p(\mathbb{R})$-valued random elements μ', μ'' satisfying $\mathrm{Law}(\mu') = L'$ and $\mathrm{Law}(\mu'') = L''$. To indicate a specific used value of p, we write $W_{1,p}$ for the distance on $\mathcal{M}_1\big(\mathcal{M}_p(\mathbb{R})\big)$. The main result of the second section is the following theorem {cf. [17] (Theorem 1), [4] (Theorem 1.3)}:

Theorem 1.1. *Assume that the sequence $\{n\delta_n\}_{n\in\mathbb{N}}$ is bounded. Then, for every $p \geq 2$, there exist a positive constant C and a number $N \in \mathbb{N}$ such that, for all $n \geq N$,*

$$W_{1,p}\big(L_t, L_t^{(n)}\big) \leq C\big(\log \delta_n^{-1}\big)^{-1/p}.$$

Section 3 is devoted to the counting measure associated with the Arratia flow. We discuss the rate of convergence of these measures by approximating a segment of the real line by its finite subsets. For this purpose, we introduce multidimensional densities corresponding to different sequences of collisions in the n-point motion of the Arratia flow.

Given an Arratia flow $\{X(u,t) \mid t \geq 0,\ u \in [0;1]\}$ with zero drift, we set

$$\Delta_n = \{u_1 < \ldots < u_n\}, \quad n \in \mathbb{N}, \qquad \text{and} \qquad X_t = \{X(u,t) \mid u \in [0;1]\}.$$

The next definition is taken from [14] (Appendix B) (see also [7, 15]) and adjusted to reflect the fact that the Arratia flow now starts from $[0;1]$ instead of the whole real line.

Definition 1.1. *The n-point density p_t^n is a measurable function such that, for any bounded nonnegative measurable $f\colon \mathbb{R}^n \to \mathbb{R}$, we have*

$$\int\limits_{\mathbb{R}^n} f(x)p_t^n(x)\,dx = \mathrm{E} \sum_{\substack{u_1,\ldots,u_n \in X_t \\ \textit{all distinct}}} f(u_1,\ldots,u_n). \tag{1.2}$$

Recall that, given $u = (u_1,\ldots,u_n)$, the processes $X(u_1),\ldots,X(u_n)$ are coalescing Brownian motions. To describe all possible sequences of collisions in this system, we use the following notation:

We define $\mathcal{X}^n \in \big(C([0;1])\big)^n$ by setting $\mathcal{X}_j^n(\cdot) = X(u_j,\cdot),\ j = \overline{1,n}$. Let k be the number of distinct values in the set $\big\{X(u_1,t),\ldots,X(u_n,t)\big\}$. Suppose that $k < n$. Let τ_1 be the time of the first collision on $[0;t]$. We set

$$j_1 = \min\big\{i \mid \exists j \neq i\ \mathcal{X}_i^n(\tau_1) = \mathcal{X}_j^n(\tau_1)\big\}$$

and define $\mathcal{X}^{n-1} \in \big(C([0;1])\big)^{n-1}$ by excluding the j_1th coordinate from $\mathcal{X}^n$. If there exists a moment $\tau_2 \leq t$ such that $\mathcal{X}_i^{n-1}(\tau_2) = \mathcal{X}_j^{n-1}(\tau_2)$ for some $i,j \in \{1,\ldots,n-1\}$, then we set j_2 equal to the smallest number of this kind. Repeating the procedure $n-k$ times, we obtain a random collection

$$J_t(u) = (j_1,\ldots,j_{n-k}), \quad j_i \in \big\{1,\ldots,n-i\big\}, \quad i = \overline{1,n-k}.$$

In the case $k = n$, we set $J_t(u) = \varnothing$ by definition. The set of all possible collections of l numbers of this kind is denoted by $\mathcal{J}_{n,l}$.

Definition 1.2. *The random collection $J_t(u)$ defined via the recursive procedure described above is called the coalescence scheme corresponding to the start points $u_1,\ldots,u_n$.*

Definition 1.3. *Given $x = (x_1, \dots, x_n) \in \Delta_n$, the k-point density $p_t^{J,k}(x;\cdot)$ corresponding to the coalescence scheme $J \in \mathcal{J}_{n,n-l}$, $k \le l$, and the start points $x_1, \dots, x_n$ is a measurable function such that, for any bounded nonnegative measurable $f\colon \mathbb{R}^k \to \mathbb{R}$,*

$$\int\limits_{\mathbb{R}^k} p_t^{J,k}(x;y) f(y)\, dy = \mathrm{E} \sum_{\substack{u_1,\dots,u_k \in \{X(x_1,t),\dots,X(x_n,t)\} \\ \textit{all distinct}}} f(u_1,\dots,u_k) \times \mathbb{1}(J_t(x) = J). \tag{1.3}$$

The integral representation for these densities is obtained (see Theorem 3.1). The result on convergence of the multidimensional densities given in Theorem 3.2 is motivated by the discrete approximations from Section 2.

Consider the vectors $U^{(n)} = \bigl(u_1^{(n)}, \dots, u_n^{(n)}\bigr) \in \Delta^n$ such that

$$u_1^{(n)} = 0, \quad u_n^{(n)} = 1, \quad n \in \mathbb{N},$$

$$\limsup_{n\to\infty} \max_{j=\overline{0,n-1}} \Bigl(u_{j+1}^{(n)} - u_j^{(n)}\Bigr) = 0,$$

and

$$\bigl\{u_1^{(n)}, \dots, u_n^{(n)}\bigr\} \subset \bigl\{u_1^{(n+1)}, \dots, u_{n+1}^{(n+1)}\bigr\}, \quad n \in \mathbb{N}.$$

Define

$$p_t^k(U^{(n)};\cdot) = \sum_{i=k}^{n} \sum_{J \in \mathcal{J}_{n,n-i}} p_t^{J,k}(U^{(n)};\cdot), \qquad k = \overline{1,n}, \quad n \in \mathbb{N}. \tag{1.4}$$

Theorem 1.2. *There exists an absolute positive constant C such that*

$$0 \le p_t^1(y) - p_t^1\bigl(U^{(n)};y\bigr) \le C \max_{j=\overline{1,n-1}} \Bigl(u_{j+1}^{(n)} - u_j^{(n)}\Bigr)^2$$

for almost all y.

2. The Wasserstein Distance Between L_t and $L_t^{(n)}$

We approximate the measures μ_t and $\mu_t^{(n)}$ by point measures

$$\mu_t^{(n),m} = m^{-1} \sum_{j=0}^{m-1} \delta_{\Phi_t^{(n)}(j/m)},$$

$$\mu_t^m = m^{-1} \sum_{j=0}^{m-1} \delta_{\Phi_t^a(j/m)}, \quad n, m \in \mathbb{N}.$$

We begin with L_p-estimates for the divergence between two solutions of a one-dimensional SDE in terms of the difference between the initial points and estimates of the same type for their approximations by the fractional step method.

Let a be a bounded function satisfying the Lipschitz condition with a constant C_a. We set $M_a = \sup_{\mathbb{R}} |a|$. Given a standard Brownian motion w and a point $u \in \mathbb{R}$, the equation

$$dx(t) = a(x(t))dt + dw(t),$$

$$x(0) = u, \quad t \in [0;1],$$

has the unique strong solution x. For $t \in \Delta_j^{(n)},\ j = \overline{0, n-1}$, we consider

$$\begin{aligned} y^{(n)}(t) &= u + \int\limits_0^{t_{j+1}^{(n)}} a\big(z^{(n)}(s)\big)\, ds + w(t), \\ z^{(n)}(t) &= u + \int\limits_0^{t} a\big(z^{(n)}(s)\big)\, ds + w\big(t_j^{(n)}\big). \end{aligned} \tag{2.1}$$

We encode the relations between $x,\ y^{(n)},\ z^{(n)}$ and $w,\ u$ by writing

$$x = D(w,u), \qquad (y^{(n)}, z^{(n)}) = S^{(n)}(w,u).$$

The next result is a straightforward generalization of [1] (Corollary 4.2).

Lemma 2.1. *For any $p \geq 1$, there exists $C > 0$ such that*

$$\mathrm{E} \sup_{s \leq 1} \big|x(s) - y^{(n)}(s)\big|^p \leq C\delta_n^{p/2},$$

$$\sup_{s \leq 1} \mathrm{E} \Big|x(s) - z^{(n)}(s)\Big|^p \leq C\delta_n^{p/2}.$$

Lemma 2.2. *Suppose that $u_1, u_2 \in \mathbb{R}$, and $w_1,\ w_2$ are independent Brownian motions. Let $x_k = D(w_k, u_k)$, $k = 1, 2$. Then, for any $p \geq 1$, there exists $C > 0$ such that*

$$\mathrm{E} \sup_{s \leq 1} \big|x_1(s \wedge \theta) - x_2(s \wedge \theta)\big|^p \leq C\big(|u_1 - u_2| + |u_1 - u_2|^p\big), \quad p \geq 2,$$

$$\mathrm{E} \sup_{s \leq 1} \big|x_1(s \wedge \theta) - x_2(s \wedge \theta)\big|^p \leq C\big(|u_1 - u_2|^{p/2} + |u_1 - u_2|^p\big), \quad p \in [1;2),$$

where $\theta = \inf\big\{1; s \mid x_1(s) = x_2(s)\big\}$.

Proof. Denote $\Delta u = u_2 - u_1$ and $\Delta x = x_2 - x_1$. Assume that $u_2 > u_1$. Consider a SDE

$$d\eta(t) = C_a \eta(t) dt + dw_2(t) - dw_1(t),$$

$$\eta(0) = \Delta u,$$

with the unique strong solution

$$\eta(t) = e^{C_a t}\Delta u + \sqrt{2}e^{C_a t}\int\limits_0^t e^{-C_a s}\,dw(s), \tag{2.2}$$

where $w = \dfrac{w_2 - w_1}{\sqrt{2}}$. We have

$$\eta(t) - \Delta x(t) = C_a\int\limits_0^t \big(\eta(s) - \Delta x(s)\big)\,ds$$

$$+\int\limits_0^t \big(C_a\Delta x(s) - a(x_2(s)) + a(x_1(s))\big)\,ds \quad \text{a.s..}$$

Therefore, a.s.,

$$\eta(t) - \Delta x(t) = e^{C_a t}\int\limits_0^t e^{-C_a s}\big(C_a\Delta x(s) - a(x_2(s)) + a(x_1(s))\big)\,ds \geq 0, \quad t \in [0;\theta]. \tag{2.3}$$

Applying the Knight theorem [13] (Proposition 18.8) to the stochastic integral in (2.2), we get

$$\eta(t) = e^{C_a t}\Delta u + \sqrt{2}e^{C_a t}\beta\left(\int\limits_0^t e^{-2C_a s}ds\right),$$

where β is a Brownian motion. Then (2.3) implies that

$$\theta \leq \varkappa = \inf\{1; s \mid \eta(s) = 0\} = \inf\left\{1; s \;\middle|\; \beta\left(\int\limits_0^t e^{-2C_a s}ds\right) = \frac{-\Delta u}{\sqrt{2}}\right\}.$$

Thus,

$$\mathrm{E}\sup_{t\leq\theta}|\Delta x(t)|^p \leq \mathrm{E}\sup_{t\leq\varkappa}|\eta(t)|^p \leq 2^{p-1}e^{pC_a}\Delta u^p + 2^{3p/2-1}e^{pC_a}\,\mathrm{E}\sup_{t\leq\varkappa}|\beta(t)|^p,$$

since

$$\int\limits_0^t e^{-2C_a s}ds < t, \quad t \geq 0.$$

The same reason implies that the random moment $\varkappa$ is a stopping time w.r.t. the filtration generated by $\big\{\beta(t) \mid t \in [0;1]\big\}$. Therefore, by the Burkholder–Davis–Gundy inequality, we get

$$\mathrm{E}\sup_{t\leq\varkappa}|\beta(t)|^p \leq C_p\,\mathrm{E}\,\varkappa^{p/2}, \quad p \geq 2,$$

for positive constants C_p. The distribution of $\varkappa$ is given by

$$\mathsf{P}(\varkappa \geq t) = \sqrt{\frac{2}{\pi}} \int\limits_0^{a(t)} e^{-y^2/2} dy,$$

$$a(t) = \frac{C_a^{1/2}(u_2 - u_1)}{(1 - e^{-2t})^{1/2}}.$$

Hence, for fixed $p \geq 2$,

$$\mathrm{E}\varkappa^{p/2} = \frac{p}{2} \int\limits_0^1 t^{\frac{p}{2}-1} \left(\sqrt{\frac{2}{\pi}} \int\limits_0^{a(t)} e^{-y^2/2} dy \right) dt \leq \frac{p}{\sqrt{2\pi}} \int\limits_0^1 a(t) t^{p/2-1} dt \leq C(u_2 - u_1) \tag{2.4}$$

for some C. To handle the case $p \in [1; 2)$, it is necessary to use the Lyapunov inequality and the previous estimates.

Lemma 2.2 is proved.

We now consider a modification of (2.1): On every $\Delta_j^{(n)},\ j = \overline{0, n-1}$,

$$y^{(n)}(t) = u_y + \int\limits_0^{t_{j+1}^{(n)}} a\big(z^{(n)}(s)\big)\, ds + w(t),$$

$$z^{(n)}(t) = u_z + \int\limits_0^{t} a\big(z^{(n)}(s)\big)\, ds + w(t_j^{(n)}), \quad t \in \Delta_j^{(n)},$$

where nonrandom u_y and u_z are not necessarily equal. The pair $(y^{(n)}, z^{(n)})$ is denoted by $S^{(n)}(w, u_y, u_z)$.

Lemma 2.3. *Assume that the sequence $\{n\delta_n\}_{n\in\mathbb{N}}$ is bounded. Let $u_{y_1}, u_{y_2}, u_{z_1}, u_{z_2} \in \mathbb{R}$ and let $w_1,\ w_2$ be independent standard Brownian motions. Also let*

$$\big(y_k^{(n)}, z_k^{(n)}\big) = S^{(n)}(w_k, u_{y_k}, u_{z_k}), \quad k = 1, 2.$$

Then, for any $p \geq 2$ and any $\varepsilon \in \left(0; \frac{1}{2}\right)$, there exist $C > 0$ and $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\mathrm{E} \sup_{s \leq 1} \left| y_1^{(n)}(s \wedge \theta^{(n)}) - y_2^{(n)}(s \wedge \theta^{(n)}) \right|^p$$

$$\leq C \left(\delta_n^{1/2-\varepsilon} + \sum_{l=1}^{2} \big(|u_{z_l} - u_{y_l}| + |u_{z_l} - u_{y_l}|^p \big) + |u_{y_2} - u_{y_1}|^p + |u_{y_2} - u_{y_1}| \right),$$

where $\theta^{(n)} = \inf\big\{1; s \mid y_2^{(n)}(s) = y_1^{(n)}(s)\big\}$.

Proof. We extend the proof of Lemma 2.2. Suppose that $u_{z_2} - u_{z_1} \geq 0,\ u_{y_2} - u_{y_1} \geq 0$. Denote

$$\Delta u_y = u_{y_2} - u_{y_1} \qquad \text{and} \qquad \Delta u_z = u_{z_2} - u_{z_1}.$$

Further, let η be defined as in (2.2) with $\Delta u = \Delta u_y$. Then, for $t \leq \theta^{(n)},\ t \in \Delta_j^{(n)}$, for some j, and for $\Delta y = y_2 - y_1$, we have

$$\begin{aligned}
\Delta y(t) - \eta(t) &= C_a \int_0^t (\Delta y(s) - \eta(s))\, ds \\
&\quad + \int_0^t \Big(a(z_2^{(n)}(s)) - a(z_1^{(n)}(s)) - C_a \Delta y(s) \Big) ds \\
&\quad + \int_t^{t_{j+1}^{(n)}} \Big(a(z_2^{(n)}(s)) - a(z_1^{(n)}(s)) \Big) ds \\
&\leq C_a \int_0^t (\Delta y(s) - \eta(s))\, ds + C_a \int_0^t \sum_{l=1}^{2} (-1)^l \big(z_l^{(n)}(s) - y_l^{(n)}(s) \big)\, ds + 2\delta_n M_a,
\end{aligned}$$

since $z_2^{(n)} \geq z_1^{(n)}$ on $[0; \theta^{(n)}]$. For $s \in \Delta_i^{(n)},\ i \leq j$, we find

$$\begin{aligned}
\Big| z_k^{(n)}(s) - y_k^{(n)}(s) - w_k(t_i^{(n)}) + w_k(s) \Big| &\leq \int_s^{t_{i+1}^{(n)}} \big| a\big(z_k^{(n)}(s)\big) \big| ds + |u_{z_k} - u_{y_k}| \\
&\leq \big(t_{i+1}^{(n)} - s \big) M_a + |u_{z_k} - u_{y_k}|.
\end{aligned}$$

Hence, for $t \in \Delta_j^{(n)},\ t \leq \theta^{(n)}$, this yields

$$\begin{aligned}
\Delta y(t) - \eta(t) &\leq C_a \int_0^t (\Delta y(s) - \eta(s))\, ds \\
&\quad + C_a \sum_{k=0}^{j-1} \int_{\Delta_k^{(n)}} \sum_{l=1}^{2} (-1)^l \big(w_l(t_{k+1}^{(n)}) - w_l(s) \big)\, ds \\
&\quad + 2C_a M_a \sum_{k=0}^{j-1} \int_{\Delta_k^{(n)}} (t_{k+1}^{(n)} - s)\, ds + 2\delta_n M_a + \sum_{l=1}^{2} |u_{z_l} - u_{y_l}|.
\end{aligned}$$

Since

$$2C_aM_a\sum_{k=0}^{j-1}\int\limits_{\Delta_k^{(n)}}\left(t_{k+1}^{(n)}-s\right)ds\le C_aM_a\delta_n,$$

the Gronwall–Bellman inequality implies that

$$\Delta y(t)\le\eta(t)+e^{C_a}M_a(C_a+2)\delta_n+e^{C_a}\sum_{l=1}^{2}|u_{z_l}-u_{y_l}|+e^{C_a}C_a\max_{j=\overline{1,n}}|\xi_j|\,,$$

where

$$\xi_j=\sum_{l=1}^{2}\sum_{k=0}^{j-1}\int\limits_{\Delta_k^{(n)}}(-1)^l\big(w_l(t_{k+1}^{(n)})-w_l(s)\big)\,ds,\quad j=\overline{1,n}.$$

Thus,

$$\begin{aligned}\mathrm{E}\sup_{s\le\theta^{(n)}}|\Delta y(s)|^p\le 4^{p-1}\Bigg(&\mathrm{E}\sup_{s\le\theta^{(n)}}|\eta(s)|^p+e^{pC_a}M_a^p(C_a+2)^p\delta_n^p\\&+e^{pC_a}\left(\sum_{l=1}^{2}|u_{z_l}-u_{y_l}|\right)^p+e^{pC_a}C_a^p\,\mathrm{E}\max_{j=\overline{1,n}}|\xi_j|^p\Bigg).\end{aligned}\tag{2.5}$$

The random variables $\xi_{j+1}-\xi_j,\ j=\overline{1,n-1},$ are independent centered Gaussian variables and

$$\mathrm{Var}(\xi_n)\le 2n\delta_n^2.$$

Therefore, by the Levy inequality, there exists a constant C such that

$$\mathrm{E}\max_{j=\overline{1,n}}|\xi_j|^p\le 2\,\mathrm{E}|\xi_n|^p\le Cn^p\delta_n^{2p}\tag{2.6}$$

and, for any $x_n>0,$ we get

$$\mathsf{P}\left(\max_{k=\overline{1,n}}|\xi_k|\ge x_n\right)\le 2\,\mathsf{P}\left(\big|\mathbb{N}(0,1)\big|\ge\frac{x_n}{(\mathrm{Var}(\xi_n))^{1/2}}\right)\le C\frac{n^{1/2}\delta_n}{x_n}e^{-\frac{x_n^2}{4n\delta_n^2}}.\tag{2.7}$$

At the same time, proceeding exactly as in the proof of Lemma 2.2 we obtain

$$\mathrm{E}\sup_{t\le\theta^{(n)}}\big|\eta(t)\big|^p\le 2^{p-1}e^{pC_a}\Delta u_y^p+2^{3p/2-1}e^{pC_a}C_p\mathrm{E}(\theta^{(n)})^{p/2}.\tag{2.8}$$

However, at time $\theta^{(n)}$,

$$\eta(\theta^{(n)})\ge -e^{C_a}M_a(C_a+2)\delta_n-e^{C_a}\sum_{l=1}^{2}|u_{z_l}-u_{y_l}|-e^{C_a}C_a\max_{j=\overline{1,n}}|\xi(j)|.$$

Hence, for fixed $x_n > 0$,

$$\mathrm{E}(\theta^{(n)})^{p/2} \le \mathsf{P}\left(\max_{k=\overline{1,n}} |\xi_k| \ge x_n\right) + \mathrm{E}\tau_n^{p/2},$$

where

$$\tau_n = \inf\left\{1;\ \ s \,\middle|\, \eta(s) = -e^{C_a} M_a(C_a+2)\delta_n - e^{C_a}\sum_{l=1}^{2} |u_{z_l} - u_{y_l}| - e^{C_a} C_a x_n\right\}.$$

We set $K = \sup_{k\in\mathbb{N}} k\delta_k$. Reasoning leading to (2.4), in combination with (2.7), implies that

$$\mathrm{E}(\theta^{(n)})^{p/2} \le C\left(\Delta u_y + \delta_n + \sum_{l=1}^{2} |u_{z_l} - u_{y_l}| + x_n + K^{1/2}\frac{\delta_n^{1/2}}{x_n} e^{-\frac{x_n^2}{4K\delta_n}}\right) \tag{2.9}$$

for the redefined constant C. Choosing $x_n = \delta_n^{1/2-\varepsilon}$, for any fixed $\varepsilon \in \left(0; \frac{1}{2}\right)$, and substituting (2.6), (2.8), and (2.9) in (2.5), we complete the proof.

We now recall the definitions of the considered measures. For the random elements in $\mathcal{M}_p(\mathbb{R})$

$$\mu_t = \lambda \circ \left(\Phi_t^a\right)^{-1}, \qquad \mu_t^m = \left(\frac{1}{m}\sum_{j=1}^{m} \delta_{j/m}\right) \circ \left(\Phi_t^a\right)^{-1},$$

$$\mu_t^{(n)} = \lambda \circ \left(\Phi_t^{(n)}\right)^{-1}, \qquad \mu_t^{(n),m} = \left(\frac{1}{m}\sum_{j=1}^{m} \delta_{j/m}\right) \circ \left(\Phi_t^{(n)}\right)^{-1}, \quad n, m \in \mathbb{N},$$

we consider their distributions as elements of $\mathcal{M}_1\big(\mathcal{M}_p(\mathbb{R})\big)$:

$$L_t = \mathrm{Law}(\mu_t), \qquad L_t^m = \mathrm{Law}\big(\mu_t^m\big),$$

$$L_t^{(n)} = \mathrm{Law}\big(\mu_t^{(n)}\big), \qquad L_t^{(n),m} = \mathrm{Law}\big(\mu_t^{(n),m}\big), \quad n, m \in \mathbb{N}.$$

By analogy with [4] (Theorem 2.1), we get the following lemma:

Lemma 2.4. *For any $p \ge 2$ there exists $C > 0$ such that*

$$W_{1,p}\big(L_t, L_t^m\big) \le C m^{-1/p}.$$

If, in addition, $\{n\delta_n\}_{n\in\mathbb{N}}$ is bounded, then

$$W_{1,p}(L_t^{(n)}, L_t^{(n),m}) \le C\big(m^{-1} + \delta_n^{1/2-\varepsilon}\big)^{1/p}.$$

Proof. Since the random measures $\big(\mu_t, \mu_t^m\big)$ form a coupling for the pair $\big(L_t, L_t^m\big)$, it follows from the definition of the distance $W_{1,p}$ that

$$W_{1,p}\big(L_t, L_t^m\big) \le \mathrm{E} W_p\big(\mu_t, \mu_t^m\big).$$

Therefore, by [16] (Theorem 2.18, Remark 2.19) and Lemma 2.2, for some C, we get

$$W_{1,p}\big(L_t, L_t^m\big) \le \left(\sum_{j=0}^{m-1} \int\limits_{j/m}^{(j+1)/m} \mathrm{E}\big|\Phi_t^a(y) - \Phi_t^a(j/m)\big|^p dy\right)^{1/p}$$

$$\le C\left(\sum_{j=0}^{m-1} \int\limits_{j/m}^{(j+1)/m} (y - j/m)\, dy\right)^{1/p} = Cm^{-1/p}$$

because, for x_1 and x_2 from Lemma 2.2,

$$\Big\{\big(\Phi^a_{t\wedge\theta_1}(y), \Phi^a_{t\wedge\theta_1}(j/m)\big) \mid t \in [0;1]\Big\} \stackrel{d}{=} \Big\{\big(x_1(t\wedge\theta_2), x_2(t\wedge\theta_2)\big) \;\Big|\; t \in [0;1]\Big\},$$

where θ_1 and θ_2 are the moments of meeting for the corresponding pairs of processes. Similarly, by using Lemma 2.3 with $u_{z_k} = u_{y_k}$, $k = 1, 2$, we find

$$W_{1,p}\big(L_t^{(n)}, L_t^{(n),m}\big) \le C\left(\sum_{j=0}^{m-1} \int\limits_{j/m}^{(j+1)/m} (y - j/m)\; dy + \delta_n^{1/2-\varepsilon}\right)^{1/p} \le C\big(m^{-1} + \delta_n^{1/2-\varepsilon}\big)^{1/p}$$

for some C.

Lemma 2.4 is proved.

We now describe appropriate couplings for $\big(\mu_t^m, \mu^{(n),m}\big)$, $n \in \mathbb{N}$, and given fixed m. Suppose that $w_1, \dots, w_m$ are independent standard Brownian motions. Denoting $u_j = j/m$, $j = \overline{0, m}$, we set

$$x_j = D(w_j, u_j),$$

$$(y_j^{(n)}, z_j^{(n)}) = S^{(n)}(w_j, u_j), \quad n \in \mathbb{N},$$

and define

$$\widetilde{x}_1 = x_1, \quad \widetilde{y}_1^{(n)} = y_1^{(n)}, \quad \text{and} \quad \widetilde{z}_1^{(n)} = z_1^{(n)}.$$

Proceeding recursively, we set

$$\theta_j = \inf\big\{1; s \mid x_j(s) = \widetilde{x}_{j-1}(s)\big\},$$

$$\theta_j^{(n)} = \inf\big\{1; s \mid y_j^{(n)}(s) = \widetilde{y}_{j-1}^{(n)}(s)\big\},$$

$$\widetilde{x}_j(t) = x_j(t)\mathbb{1}\left(t < \theta_j\right) + \widetilde{x}_{j-1}(t)\mathbb{1}\left(t \geq \theta_j\right),$$

$$\widetilde{y}_j^{(n)}(t) = y_j^{(n)}(t)\mathbb{1}\big(t < \theta_j^{(n)}\big) + \widetilde{y}_{j-1}^{(n)}(t)\mathbb{1}\big(t \geq \theta_j^{(n)}\big), \quad j = \overline{2,m}.$$

Further, we consider a random number $k_j^{(n)}$ such that $\theta_j^{(n)} \in \Delta_j^{(n)}$ and set

$$\widetilde{z}_j^{(n)}(t) = z_j^{(n)}(t)\mathbb{1}\Big(t < t^{(n)}_{k_j^{(n)}+1}\Big) + \widetilde{z}_{j-1}^{(n)}(t)\mathbb{1}\Big(t \geq t^{(n)}_{k_j^{(n)}+1}\Big), \qquad t \in [0;1), \quad j = \overline{2,m}.$$

The values at $t = 1$ are taken to be equal to the corresponding left limits. The processes

$$\widetilde{w}_1 = w_1, \qquad \widetilde{w}_1^{(n)} = w_1,$$

$$\widetilde{w}_j(t) = w_j(t)\mathbb{1}\left(t < \theta_j\right) + \widetilde{w}_{j-1}(t)\mathbb{1}\left(t \geq \theta_j\right),$$

$$\widetilde{w}_j^{(n)}(t) = w_j(t)\mathbb{1}\big(t < \theta_j^{(n)}\big) + \widetilde{w}_{j-1}^{(n)}(t)\mathbb{1}\big(t \geq \theta_j^{(n)}\big), \qquad j = \overline{2,m}, \quad n \in \mathbb{N},$$

can be checked to be Brownian motions.

The proofs of the following two lemmas are based on the repeated application of (2.1) and, hence, are omitted.

Lemma 2.5. *For $n, m \in \mathbb{N}$ and $j = \overline{1,m}$,*

$$\widetilde{x}_j = D(\widetilde{w}_j, u_j),$$

$$\big(\widetilde{y}_j^{(n)}, \widetilde{z}_j^{(n)}\big) = S^{(n)}(\widetilde{w}_j^{(n)}, u_j).$$

Lemma 2.6. *For $n, m \in \mathbb{N}$,*

$$\big(\Phi^a(u_1), \ldots, \Phi^a(u_m)\big) \stackrel{d}{=} \big(\widetilde{x}_1, \ldots, \widetilde{x}_m\big),$$

$$\Big(\Phi_{0,\cdot}^{(n)}(u_1), \ldots, \Phi_{0,\cdot}^{(n)}(u_m)\Big) \stackrel{d}{=} \big(\widetilde{y}_1^{(n)}, \ldots, \widetilde{y}_m^{(n)}\big)$$

in $\big(D([0;1])\big)^m$.

Proof of Theorem 1.1. Repeating the reasoning from the proof of Lemma 2.4 and using Lemma 2.6, we get

$$\big(W_{1,p}\big(L_t^m, L_t^{(n),m}\big)\big)^p \leq \sum_{j=0}^{m-1} \int\limits_{j/m}^{(j+1)/m} \mathrm{E}\big|\widetilde{x}_j(t) - \widetilde{y}_j^{(n)}(t)\big|^p du = m^{-1} \sum_{j=0}^{m-1} \mathrm{E}\big|\widetilde{x}_j(t) - \widetilde{y}_j^{(n)}(t)\big|^p.$$

By Lemma 2.1, for some positive C_1, we have

$$\mathrm{E}\big|\widetilde{x}_1(t) - \widetilde{y}_1^{(n)}(t)\big|^p \leq \mathrm{E} \sup_{s \leq 1} \big|x_1(s) - y_1^{(n)}(s)\big|^p \leq C_1 \delta_n^{p/2}.$$

Continuing, for $j = 2$, we find

$$\begin{aligned}
\mathrm{E}\big|\widetilde{x}_2(t) - \widetilde{y}_2^{(n)}(t)\big|^p &= \mathrm{E}\big|\widetilde{x}_2(t) - \widetilde{y}_2^{(n)}(t)\big|^p \times \Big[\mathbb{1}\big(t \ge \theta_1 \wedge \theta_1^{(n)}\big) + \mathbb{1}\big(\theta_1^{(n)} \le t < \theta_1\big) \\
&\qquad + \mathbb{1}\big(\theta_1 \le t < \theta_1^{(n)}\big) + \mathbb{1}\big(t < \theta_1^{(n)} \wedge \theta_1\big)\Big] \\
&\le \mathrm{E}\big|\widetilde{x}_1(t) - \widetilde{y}_1^{(n)}(t)\big|^p \mathbb{1}\big(t \ge \theta_1 \wedge \theta_1^{(n)}\big) + \mathrm{E}\left|x_2(t) - y_2^{(n)}(t)\right|^p \mathbb{1}\big(t < \theta_1^{(n)} \wedge \theta_1\big) \\
&\qquad + 2^{p-1}\mathrm{E}\Big[\big|x_2(t) - \widetilde{x}_1(t)\big|^p + \big|\widetilde{x}_1(t) - \widetilde{y}_1^{(n)}(t)\big|^p\Big]\mathbb{1}\big(\theta_1^{(n)} \le t < \theta_1\big) \\
&\qquad + 2^{p-1}\mathrm{E}\Big[\big|\widetilde{x}_1(t) - \widetilde{y}_1^{(n)}(t)\big|^p + \big|\widetilde{y}_1^{(n)}(t) - y_2^{(n)}(t)\big|^p\Big]\mathbb{1}\big(\theta_1 \le t < \theta_1^{(n)}\big) \\
&\le 2^{p-1}\mathrm{E}\big|\widetilde{x}_1(t) - \widetilde{y}_1^{(n)}(t)\big|^p + 2^{p-1}\mathrm{E}\big|\widetilde{x}_1(t) - x_2(t)\big|^p \mathbb{1}\big(\theta_1^{(n)} \le t < \theta_1\big) \\
&\qquad + 2^{p-1}\mathrm{E}\big|\widetilde{y}_1^{(n)}(t) - y_2^{(n)}(t)\big|^p \mathbb{1}\big(\theta_1 \le t < \theta_1^{(n)}\big) + \mathrm{E}\big|x_2(t) - y_2^{(n)}(t)\big|^p.
\end{aligned}$$

By using Lemma 2.1 once again, we obtain

$$\begin{aligned}
\mathrm{E}\big|\widetilde{x}_2(t) - \widetilde{y}_2^{(n)}(t)\big|^p &\le (2^{p-1}+1)C_1\delta_n^{p/2} \\
&\qquad + 2^{p-1}\mathrm{E}\sup_{\theta_1^{(n)} \le s \le \theta_1}\big|\widetilde{x}_1(s) - x_2(s)\big|^p \mathbb{1}\big(\theta_1^{(n)} \le \theta_1\big) \\
&\qquad + 2^{p-1}\mathrm{E}\sup_{\theta_1 \le s \le \theta_1^{(n)}}\big|\widetilde{y}_1^{(n)}(s) - y_2^{(n)}(s)\big|^p \mathbb{1}\big(\theta_1 \le \theta_1^{(n)}\big). \qquad (2.10)
\end{aligned}$$

We consider the last two terms in (2.10) separately. Note that $\widetilde{x}_1$ and x_2 are independent and the same is true for $\widetilde{y}_1$ and y_2. Thus, by using the Markov property, we can show that

$$\mathrm{E}\sup_{\theta_1^{(n)} \le s \le \theta_1}\big|\widetilde{x}_1(s) - x_2(s)\big|^p \mathbb{1}\big(\theta_1^{(n)} \le \theta_1\big) \le \mathrm{E}\sup_{0 \le s \le \tau_1}\big|\eta_1(s) - \eta_2(s)\big|^p, \qquad (2.11)$$

where $\eta_k = D(\beta_k, v_k)$, $k = 1, 2$, with independent Brownian motions β_1 and β_2 also independent of w_1 and w_2 (and, therefore, of $\widetilde{x}_1$ and x_2), and

$$v_1 = \widetilde{x}_1\big(\theta_1^{(n)}\big), \qquad v_2 = x_2\big(\theta_1^{(n)}\big),$$

$$\tau_1 = \inf\big\{1; s \mid \eta_1(s) = \eta_2(s)\big\}.$$

Thus, by the first inequality in Lemma 2.1, for any $q \ge 1$,

$$\mathrm{E}\,|v_1 - v_2|^q \le \mathrm{E}\left|\widetilde{x}_1\big(\theta_1^{(n)}\big) - \widetilde{y}_1^{(n)}\big(\theta_1^{(n)}\big)\right|^q + \mathrm{E}\left|x_2\big(\theta_1^{(n)}\big) - y_2^{(n)}\big(\theta_1^{(n)}\big)\right|^q \le 2C_1\delta_n^{q/2}.$$

Hence, by taking the conditional expectation in (2.11) and averaging over v_1 and v_2, in view of Lemma 2.2, we get

$$\mathrm{E} \sup_{\theta_1^{(n)} \le s \le \theta_1} \big|\widetilde{x}_1(s) - x_2(s)\big|^p \mathbb{1}\big(\theta_1^{(n)} \le \theta_1\big) \le C_2 \delta_n^{1/2} \tag{2.12}$$

for some C_2. Similarly,

$$\mathrm{E} \sup_{\theta_1 \le s \le \theta_1^{(n)}} \big|\widetilde{y}_1^{(n)}(s) - y_2^{(n)}(s)\big|^p \mathbb{1}\big(\theta_1 \le \theta_1^{(n)}\big) \le \mathrm{E} \sup_{0 \le s \le \tau_2} \big|\xi_1(s) - \xi_2(s)\big|^p,$$

where

$$\xi_k = \Big(S^{(n)}(\beta_k, v_{k1}, v_{k2})\Big)_1, \quad k = 1, 2,$$

and

$$v_{11} = \widetilde{y}_1^{(n)}(\theta_1), \qquad v_{12} = \widetilde{z}_1^{(n)}(\theta_1), \qquad v_{21} = y_2^{(n)}(\theta_1), \qquad v_{22} = z_2^{(n)}(\theta_1),$$

$$\tau_2 = \inf\big\{1; s \mid \xi_1(s) = \xi_2(s)\big\}.$$

By using both inequalities from Lemma 2.1, applying Lemma 2.3 with $u_{y_1} = v_{11}$, $u_{y_2} = v_{21}$, $u_{z_1} = v_{12}$, and $u_{z_2} = v_{22}$ and taking expectation, we can show that, for some positive C_3,

$$\mathrm{E} \sup_{\theta_1 \le s \le \theta_1^{(n)}} \big|\widetilde{y}_1^{(n)}(s) - y_2(s)\big|^p \mathbb{1}\big(\theta_1 \le \theta_1^{(n)}\big) \le C_3 \delta_n^{1/2-\varepsilon}. \tag{2.13}$$

Substituting (2.12) and (2.13) in (2.11), we find, for some C_4,

$$\mathrm{E}\big|\widetilde{x}_2(t) - \widetilde{y}_2^{(n)}(t)\big|^p \le C_4 \delta_n^{1/2-\varepsilon},$$

starting from some N independent of m. By using this estimate recursively for $j = 3, \dots, m$ we finally conclude that

$$\sum_{j=0}^{m-1} \mathrm{E}\big|\widetilde{x}_j(t) - \widetilde{y}_j^{(n)}(t)\big|^p \le \sum_{j=1}^{m-1} C_4^j \delta_n^{1/2-\varepsilon}.$$

By Lemma 2.4 there exist positive C_5 and a number $N' \ge N$ such that, for any $n \ge N'$,

$$W_{1,p}\big(L_t, L_t^{(n)}\big) \le C_5\big(m^{-1} + \delta_n^{1/2-\varepsilon}\big)^{1/p} + C_5\big(C_4^m \delta_n^{1/2-\varepsilon}\big)^{1/p}.$$

Therefore, choosing $m = m(n)$ such that

$$m(n) = \left(\frac{1}{4} - \frac{\varepsilon}{2}\right) \frac{\log \delta_n^{-1}}{\log C_4},$$

we complete the proof.

3. On Counting Measures Associated with the Arratia Flow

Recall that

$$\Delta_n = \{u_1 < \ldots < u_n\}, \quad n \in \mathbb{N}, \qquad \text{and} \qquad \{X(u,t) \mid t \geq 0,\ u \in [0;1]\}$$

is an Arratia flow with zero drift. Denote the density of a standard m-dimensional Brownian motion killed upon exiting Δ_m by $p^m_{0,t}$. This density is expressed via the Karlin–McGregor determinant:

$$p^m_{0,t}(x;y) = \det \big\|g_t(x_i - y_j)\big\|_{i,j=\overline{1,m}}, \quad x, y \in \Delta_m,$$

where

$$g_t(a) = \frac{1}{\sqrt{2\pi t}} e^{-a^2/2t}.$$

Any $J = (j_1, \ldots, j_{n-k}) \in \mathcal{J}_{n,n-k}$ can be associated with a partition of the set $\{1, \ldots, n\}$ by the following procedure: Starting from the partition formed by singletons, in each step $i = 1, \ldots, n-k$, we proceed by merging two subsequent blocks in the current partition with the numbers j_i and $j_i + 1$; the blocks are listed in the order of appearance w.r.t. the usual ordering of $\mathbb{N}$. The resulting partition is denoted by $\pi(J)$ and the blocks of $\pi(J)$ by $\pi_1(J), \ldots, \pi_k(J)$. Note that

$$\big\{J_t(u) = J\big\} = \Big\{\forall j \in \pi_i(J)\ \ X(x_j,t) = X(x_{\min \pi_i(J)}, t),\ \ i = \overline{1,k}\Big\}.$$

Lemma 3.1. *For all $t \in [0;1]$, $x \in \Delta_n$, $k \in \{1, \ldots, n\}$, and $J = (j_1, \ldots, j_{n-m}) \in \mathcal{J}_{n,n-m}$, $m \geq k$, the density $p^{J,k}_t(x;\cdot)$ exists. Moreover,*

$$p^{J,k}_t(x;\cdot) \leq p^k_{0,t}(x;\cdot) \quad \textit{a.e.}$$

if $m = k$.

Proof. Suppose that $k = m$. Let A be a Borel subset of Δ_k. We define a mapping $T\colon \Delta_n \mapsto \Delta_k$ by the rule

$$T(u)_l = u_{\min \pi_l(J)}, \quad l = \overline{1,k}.$$

Then

$$\mathrm{E} \sum_{u_1,\ldots,u_k \in \{X(x_1,t),\ldots,X(x_n,t)\}} \mathbb{1}_A(u_1, \ldots, u_k) \times \mathbb{1}(J_t(x) = J)$$

$$\leq \mathrm{E}\mathbb{1}\big(T\big(X(x_1,t), \ldots, X(x_n,t)\big) \in A\big) = \int\limits_A p^k_{0,t}(x;y)\, dy.$$

The Radon–Nikodym theorem yields the claim of the lemma. The cases where A is not a subset of Δ_k and $m \neq k$ are treated similarly.

Lemma 3.1 is proved.

It is possible to deduce an explicit expression for $p_t^{J,k}$. We denote the boundary of Δ_n by $\partial\Delta_n$. Moreover, we define

$$\partial\Delta_{n,j} = \Big\{(u_1,\ldots,u_n) \;\Big|\; u_1 < \ldots < u_j = u_{j+1} < \ldots < u_n\Big\}, \quad j = \overline{1,n-1}.$$

Let $w = (w_1,\ldots,w_n)$ be a standard Brownian motion. We define

$$\Delta_n(a) = \{u_1 < \ldots < u_n \le a\}, \quad n \in \mathbb{N}.$$

Theorem 3.1. *For all $t \in [0;1]$, $J = (j_1,\ldots,j_{n-k}) \in \mathcal{J}_{n,n-k}$, and $x \in \Delta_n$, the following relation is true (a.e.):*

$$p_t^{J,k}(x;y) = \int\limits_{\Delta_{n-k}(t)} dt_1 \ldots dt_{n-k} \int\limits_{\partial\Delta_{n,j_1}} m(dz_1) \int\limits_{\partial\Delta_{n,j_2}} m(dz_2)$$

$$\ldots \int\limits_{\partial\Delta_{k+1,j_{n-k}}} m(dz_{n-k})(-1)^k 2^{-k} \frac{\partial}{\partial\nu_{z_1}} p^n_{0,t_1}(x,z_1) \frac{\partial}{\partial\nu_{z_2}} p^{n-1}_{0,t_2-t_1}\big(S^n_{j_1} z_1, z_2\big)$$

$$\ldots \frac{\partial}{\partial\nu_{z_{n-k}}} p^{k+1}_{0,t_{n-k}-t_{n-k-1}}\big(S^{k+2}_{j_{n-k-1}} z_{n-k-1}, z_{n-k}\big) p^k_{0,t-t_{n-k}}\big(S^{k+1}_{j_{n-k}} z_{n-k}, y\big),$$

where m is the surface measure on $\bigcup_{j=1}^{n-1}\partial\Delta_n$, j, the operator $\dfrac{\partial}{\partial\nu_a}$ is the outward normal derivative w.r.t. the a-variables, and the mapping $S^m_j\colon \partial\Delta_{m,j} \to \Delta_{m-1}$ is given by

$$S^m_j(u_1,\ldots,u_j,u_{j+1},u_{j+2},\ldots,u_m) = (u_1,\ldots,u_j,u_{j+2},\ldots,u_m),$$

$$j = \overline{1,m-1}, \quad m \in \mathbb{N}.$$

The proof is standard and follows the ideas from [11] (Section 3) {see also [12] (Section VII.5)}.
We now recall (1.4) and note that each $p_t^k(U^{(n)};\cdot)$ satisfies (1.2) with X_t replaced by

$$X_t^{U^{(n)}} = \big\{X\big(u_1^{(n)},t\big),\ldots,X\big(u_n^{(n)},t\big)\big\}, \qquad n \in \mathbb{N}.$$

Theorem 3.2. *For all $k \in \mathbb{N}$, $p_t^k(U^{(n)};\cdot) \nearrow p_t^k$ as $n \to \infty$, a.e.*

Proof. The restrictions imposed on $\{U^{(n)}\}_{n\ge1}$ imply that (a.e.)

$$p_t^k\big(U^{(n)};\cdot\big) \le p_t^k\big(U^{(n+1)};\cdot\big) < p_t^k, \quad n \in \mathbb{N}.$$

We set

$$q(y) = \lim_{n\to\infty} p_t^k(U^{(n)};y), \quad \text{a.e.}$$

Given a bounded continuous f, the dominated convergence theorem implies that

$$\int\limits_{\mathbb{R}^k} q(y)f(y)\,dy = \lim_{n\to\infty}\int\limits_{\mathbb{R}^k} p_t^k(U^{(n)};y)f(y)\,dy$$

$$= \lim_{n\to\infty}\mathrm{E}\sum_{\substack{u_1,\dots,u_k\in X_t^{U^{(n)}}\\ \text{all distinct}}} f(u_1,\dots,u_k)\sum_{i=k}^{n}\sum_{J\in\mathcal{J}_{n,n-i}}\mathbb{1}\big(J_t\big(U^{(n)}\big)=J\big)$$

$$= \lim_{n\to\infty}\mathrm{E}\sum_{\substack{u_1,\dots,u_k\in X_t^{U^{(n)}}\\ \text{all distinct}}} f(u_1,\dots,u_k)\mathbb{1}\big(\big|X_t^{U^{(n)}}\big|\ge k\big) = \int\limits_{\mathbb{R}^k} p_t^k(y)f(y)\,dy,$$

which proves the assertion.

Theorem 3.1 can be used to study the rate of convergence in Theorem 3.2.

Proof of Theorem 1.2. Let $A_\varepsilon=[x;x+\varepsilon]$ for some $x\in\mathbb{R}$ and any $\varepsilon\ll 1$. Consider the relation

$$0\le\int\limits_{A_\varepsilon}\big(p_t^1(y)-p_t^1\big(U^{(n)};y\big)\big)\,dy=\mathrm{E}\sum_{u\in X_t}\mathbb{1}_{A_\varepsilon}(u)-\mathrm{E}\sum_{u\in X_t^{U^{(n)}}}\mathbb{1}_{A_\varepsilon}(u).$$

By using the reasoning from [14] (Appendix B), we can prove the existence of a constant C such that

$$\left|\mathrm{E}\sum_{u\in X_t}\mathbb{1}_{A_\varepsilon}(u)-\mathsf{P}(X_t\cap A_\varepsilon\ne\varnothing)\right|\le C\varepsilon^2,$$

$$\left|\mathrm{E}\sum_{x\in X_t^{U^{(n)}}}\mathbb{1}_{A_\varepsilon}(u)-\mathsf{P}\big(X_t^{U^{(n)}}\cap A_\varepsilon\ne\varnothing\big)\right|\le C\varepsilon^2.$$

Therefore,

$$\limsup_{\varepsilon\to 0+}\varepsilon^{-1}\int\limits_{A_\varepsilon}\big(p_t^1(y)-p_t^1\big(U^{(n)};y\big)\big)\,dy$$

$$=\limsup_{\varepsilon\to 0+}\varepsilon^{-1}\Big(\mathsf{P}\big(X_t\cap A_\varepsilon\ne\varnothing\big)-\mathsf{P}\big(X_t^{U^{(n)}}\cap A_\varepsilon\ne\varnothing\big)\Big). \tag{3.1}$$

By using the notion of dual Brownian web $\big\{\widetilde{X}(u,t)\mid u\in\mathbb{R},\ t\in[0;1]\big\}$ running backward in time and its noncrossing property [15] (Section 2.2), we find

$$\mathsf{P}\big(X_t\cap A_\varepsilon\ne\varnothing\big)-\mathsf{P}\big(X_t^{U^{(n)}}\cap A_\varepsilon\ne\varnothing\big)=\mathsf{P}\Big(\forall j=\overline{1,n}\ \ X\big(u_j^{(n)},t\big)\notin A_\varepsilon,\ \ X_t\cap A_\varepsilon\ne\varnothing\Big)$$

$$\le \mathsf{P}\Big(\widetilde{X}(x+\varepsilon,t) \ne \widetilde{X}(x,t), \exists\, j \in \{1,\dots,n-1\}:$$

$$\Big(\widetilde{X}(x,t); \widetilde{X}(x+\varepsilon,t)\Big) \subset \big(u_j^{(n)}; u_{j+1}^{(n)}\big)\Big)$$

$$\le \mathrm{E}\mathbb{1}\left(X(x+\varepsilon,t) - X(x,t) \le \max_{j=\overline{1,n-1}} \big(u_{j+1}^{(n)} - u_j^{(n)}\big)\right) \mathbb{1}\big(J_t(x,x+\varepsilon) = \varnothing\big)$$

$$= \int\limits_{\mathbb{R}^2} \mathbb{1}\left(y_2 - y_1 < \max_{j=\overline{1,n-1}} \big(u_{j+1}^{(n)} - u_j^{(n)}\big)\right) p_t^{\varnothing,2}\big((x,x+\varepsilon);(y_1,y_2)\big)\, dy_1\, dy_2, \qquad (3.2)$$

since X and $\widetilde{X}$ have the same distribution. Here,

$$p_t^{\varnothing,2}(a;b) = p_{0,t}^2(a;b) = \frac{1}{2\pi t} e^{-\frac{\|a-b\|^2}{2t}} \big(1 - e^{-(b_2-b_1)(a_2-a_1)}\big).$$

Thus, there exists $C > 0$ such that

$$\text{if} \quad (y_1,y_2) \in \Delta_2 \quad \text{and} \quad y_2 - y_1 \le \delta_n, \quad \text{where} \quad \delta_n = \max_{j=\overline{1,n-1}} \big(u_{j+1}^{(n)} - u_j^{(n)}\big),$$

$$\text{then} \quad p_t^{\varnothing,2}\big((x;x+\varepsilon);(y_1,y_2)\big) \le C g_t(x-y_1) \times \varepsilon\delta_n.$$

Substituting the last estimate in (3.2) and returning to (3.1) we obtain

$$\limsup_{\varepsilon\to 0+} \varepsilon^{-1} \int\limits_{A_\varepsilon} \Big(p_t^1(y) - p_t^1\big(U^{(n)};y\big)\Big)\, dy \le C \int\limits_{\mathbb{R}} dy_1 \int\limits_{y_1}^{y_1+\delta_n} dy_2\, g_t(x-y_1)\delta_n \le C\delta_n^2$$

for new C. The application of the Lebesgue differentiation theorem completes the proof.

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