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On 1-point densities for Arratia flows with drift

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ABSTRACT

We show that if drift coefficients of Arratia flows converge in $L_1(\mathbb{R})$ or $L_\infty(\mathbb{R})$ then the 1-point densities associated with these flows converge to the density for the flow with the limit drift. The main result is proven by providing a representation of the probability of coalescence in the flow as a solution to a PDE and applying the perturbation theory.

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The Arratia flow, a continual system of coalescing Wiener processes that are independent until they meet, was introduced independently as a limit of coalescing random walks in [1], as a system of reflecting Wiener processes in [21] and as a limit of stochastic homeomorphic flows in [7]. If interpreted as a collection of particles started at 0, it is a part of the Brownian web [11]. At the same time, one can construct the Arratia flow with drift using flows of kernels defined in [17] (see [20, § 6] for a short explanation) or directly via martingale problems by adapting the method used in [15] to build coalescing stochastic flows with more general dependence between particles (see [4, Chapter 7] for this approach). Following [4, Chapter 7], we consider a modification of the Arratia flow that introduces drift affecting the motion of a particle within the flow. So by an Arratia flow $X^a \equiv \{X^a(u,t) \mid u \in \mathbb{R}, t \in \mathbb{R}_+\}$ with bounded measurable drift a we understand a collection of random variables such that

(1) for every u the process $X^a(u,\cdot)$ is an Itô process with diffusion coefficient 1 and drift a;
(2) for all $t \geq 0$ the mapping $X^a(\cdot,t)$ is monotonically increasing;
(3) for any u_1, u_2 the joint quadratic covariation of the martingale parts of $X^a(u_1,\cdot)$ and $X^a(u_2,\cdot)$ equals $(t - \inf\{s \mid X^a(u_1,s) = X^a(u_2,s)\})_+$, with $\inf\emptyset$ being equal ∞ by definition.

Theorem 1 ([4, Lemma 7.3.1]). *For any measurable and locally bounded function a the flow X^a exists and is unique in the terms of finite-dimensional distributions.*

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Remark 1: [4, Lemma 7.3.1] is stated for Lipschitz continuous drift, but, being based on the classical existence and uniqueness theorems for the martingale problem, the proof holds verbatim for any measurable locally bounded drift.

Since the set $\{X^a(u,t) \mid u \in \mathbb{R}\}$ is known to be locally finite for all $t > 0$ [4, Chapter 7], one defines for any $t > 0$ the point process $\{X^a(u,t) \mid u \in \mathbb{R}\}$. Studying of such a process can be performed using point densities (see: [18,23] for a definition and a representation in terms of Pfaffians in the case of zero drift, respectively; [5,6] for representations in the case of non-trivial drift; [10,14] for applications to the study of the above-mentioned point process). In accordance with [18, Appendix B], the following definition is adopted in the present paper: the 1-point density at time t of the point process associated with X^a is given via

$$p_t^{a,1}(u) = \lim_{\delta \to 0+} \delta^{-1} \, \mathrm{P}\left(X^a(\mathbb{R},t) \cap [u; u+\delta] \neq \emptyset\right), \tag{1}$$

where the limit exists a.e.

The well-known relation between PDEs and SDEs is used to show that $p_t^{a_n,1} \to p_t^{a,1}, n \to \infty, a.e.$ whenever $a_n \to a, n \to \infty$, in $L_1(\mathbb{R})$ or $L_\infty(\mathbb{R})$ under additional natural assumptions, which is the main result of the article. It should be noted that one may approach the problem stated utilizing the representation of $p_t^{a,1}$ given in [6, Theorem 2.1]. However, the method used in the present paper allows us to avoid a discussion of properties (e.g. uniform integrability) of certain stochastic exponentials involving Brownian bridges. An additional benefit is that the term equal to $p_t^{0,1}$ is separated from those containing drift in the resulting expression. Thus there is a possibility to obtain local estimates of the density in terms of drift, in particular, in the studies of point concentration when external force modelled with non-zero drift is present. Moreover, following the route proposed in [18, § 4, Appendixes A, B] one can estimate $n-$dimensional densities in terms of $p_t^{a,1}$ given estimates for the corresponding transitional density.

Proposition 1. *The density $p_t^{a,1}$ exists.*

Proof: Following [18, Appendix B] one considers the measure $\nu^{a,1}$ on $\mathcal{B}(\mathbb{R})$ given via

$$\nu_t^{a,1}(A) = \mathrm{E}\,|X^a(\mathbb{R},t) \cap A|, \quad A \in \mathcal{B}(\mathbb{R}),$$

where given $B \in \mathcal{B}(\mathbb{R})$ $X^a(B,t) = \{X^a(u,t) \mid u \in B\}$ by definition and $|B|$ equals the number of distinct points in B. Then

$$\mathrm{E}\,|X^a(\mathbb{R},t) \cap A| = \lim_{U \to \infty} \mathrm{E}\,|X^a([-U;U],t) \cap A|. \tag{2}$$

The Girsanov theorem for the Arratia flow [4, Theorem 7.3.1] states that given $T > 0$ and a bounded Lipshitz continuous function a the distribution of X^a as a random element in the Skorokhod space $D(\mathbb{R}, C([0;T]))$ is absolutely continuous w.r.t the distribution of X^0. To extend this result to the case of $a \in L_\infty(\mathbb{R})$, one proceeds as follows. The result of [4, Lemma 7.3.2] states that for any $t > 0$ there exists $C_t > 0$ such that

$$\mathrm{P}\left\{X^a(u,t) \neq X^a(v,t)\right\} \leq C_t(v-u), \quad u, v \in \mathbb{R}, u < v. \tag{3}$$

and is valid for $a \in L_\infty(\mathbb{R})$. Using this estimate one extends the proof of [3, Theorem 2], so the flow X^a admits a version in $D(\mathbb{R}, C([0;T]))$. Then it is left to notice that the

proof of [4, Theorem 7.3.1] uses only the standard Girsanov theorem and the estimate [4, Lemma 7.3.2]. Therefore, the Girsanov theorem for the Arratia flow holds for $a \in L_\infty(\mathbb{R})$, too.

Since $\nu_t^{0,1}$ is absolutely continuous w.r.t. the Lebesgue measure [18, Appendix B], for any $A \in \mathcal{B}(\mathbb{R})$ with $\mathrm{Leb}(A) = 0$ the assumption

$$|X^0([-U;U],t) \cap A| = 0 \quad \text{a.s., } U \geq 0,$$

implies

$$\mathrm{E}\,|X^a([-U;U],t) \cap A| = 0, \quad U \geq 0.$$

Hence by (2) there exists the density $q_t^{a,1}$ such that

$$\int_A q_t^{a,1}(x)\,\mathrm{d}x = \nu_t^{a,1}(A) = \mathrm{E} \sum_{v \in X^a(\mathbb{R},t)} \mathbb{1}_A(v), \quad A \in \mathcal{B}(\mathbb{R}).$$

Analogously one proves the existence of $n-$dimensional densities $q_t^{a,n}$ such that

$$\mathrm{E} \sum_{\substack{v_1,\ldots,v_n \in X^a(\mathbb{R},t),\\ v_1,\ldots,v_n \text{ are distinct}}} \mathbb{1}_A((v_1,\ldots,v_n)) = \int_A q_t^{a,n}(x)\,\mathrm{d}x = \nu_t^{a,n}(A), \quad A \in \mathcal{B}(\mathbb{R}^n).$$

By the Lebesgue differentiation theorem, for almost all $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$,

$$q_t^{a,n}(x) = \lim_{\delta \to 0+} \delta^{-n} \nu_t^{a,n}\left(\underset{j=\overline{1,n}}{\times} [x_k; x_k + \delta] \right).$$

In particular, for almost all x

$$q_t^{a,1}(x) = \lim_{\delta \to 0+} \delta^{-1}\,\mathrm{E}\,|X^a(\mathbb{R},t) \cap [x; x+\delta]|.$$

In order to obtain (1), one still needs to replace $|X^a(\mathbb{R},t) \cap [x;x+\delta]|$ with $\mathbb{1}[X^a(\mathbb{R},t) \cap [x;x+\delta] \neq \emptyset]$ in the last expression. The argument presented in [18, Appendix B] for $a = 0$ is applicable here.

If a is additionally assumed to be Lipschitz continuous it is shown in [20, Corollary 6.1] that the dual flow [23, Section 2.2] to the flow X^a is X^{-a}. The same proof can be extended to the case of a just bounded function a. Thus, due to the analogue of (3) for $-a$ we have, for $u, v \in \mathbb{R}$, $u < v$,

$$\begin{aligned} \sup_{x \in \mathbb{R}} \mathrm{P}\left\{X^a(x,t) \in [u;v]\right\} &\leq \mathrm{P}\left\{X^a(\mathbb{R},t) \cap [u;v] \neq \emptyset\right\} \\ &= \mathrm{P}\left\{X^{-a}(u,t) \neq X^{-a}(v,t)\right\} \\ &\leq C_t(v-u). \end{aligned}$$

Using this estimate and repeating the arguments in [18, § 4], we get

$$\mathrm{P}\left\{X^a(\mathbb{R},t) \cap [u_k, v_k] \neq \emptyset, k = \overline{1,n}\right\} \leq C_t^n \prod_{k=\overline{1,n}} (v_k - u_k),$$

$$u_1 < v_1 < u_2 < \ldots u_n < v_n,$$

$$n \in \mathbb{N}.$$

Exactly as in [18, Appendix B], we then conclude

$$\nu_t^{a,n}(A) \le C_t^n \operatorname{Leb}(A), \quad A \in \mathcal{B}(\mathbb{R}^n), n \in \mathbb{N}.$$

Thus

$$\begin{aligned}
&\lim_{\delta\to 0+} \delta^{-1} \operatorname{E}\left(|X^a(\mathbb{R},t) \cap [x;x+\delta]| - \mathbb{1}\left[X^a(\mathbb{R},t) \cap [x;x+\delta] \neq \emptyset\right]\right) \\
&\quad \le \lim_{\delta\to 0+} \delta^{-1} \operatorname{E} |X^a(\mathbb{R},t) \cap [x;x+\delta]| \left(|X^a(\mathbb{R},t) \cap [x;x+\delta]| - 1\right) \\
&\quad = \lim_{\delta\to 0+} \delta^{-1} \nu_t^{a,2}\left([x;x+\delta] \times [x;x+\delta]\right) \\
&\quad \le C_t^2 \lim_{\delta\to 0+} \delta \\
&\quad = 0,
\end{aligned}$$

which concludes the proof. ■

Remark 2: If a is additionally Lipschitz continuous one can prove that $p_t^{a,n}$ has a continuous version, using the aforementioned representations from Ref. [6] or the analogue of [5, Theorem 3.1]. The same conclusion for $a \in L_\infty(\mathbb{R})$ follows from Theorem 3.

Let $D_2 = \{u \in \mathbb{R}^2 \mid u_1 < u_2\}$, and put for $a \in L_\infty(\mathbb{R})$

$$\nabla_x^a = \sum_{k=1}^{2} a(x_k)\partial_{x_k}, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

In what follows, we construct a series $W^a \in C(\overline{D}_2 \times (0;\infty)) \cap C(D_2 \times [0;\infty)) \cap C^{1,0}(D_2 \times [0;\infty))$ such that W^a is a distribution solution [9, p. 31] in $\mathcal{D}'(D_2 \times (0;t))$, the space of (Schwartz) distributions over infinitely differentiable compactly supported in $D_2 \times (0;t)$ test functions, to

$$\partial_s W^a = \frac{1}{2}\Delta W^a - \nabla_x^a W^a. \tag{4}$$

Additionally, as a function, W^a satisfies

$$\begin{aligned}
W^a(x,0) &= 1, \quad x \in D_2, \\
W^a(x,s) &= 0, \quad x \in \partial D_2,\ s > 0.
\end{aligned} \tag{5}$$

The density $p_t^{a,1}$ admits a representation as a derivative of W^a. As a result, in order to prove the convergence of the 1-point densities it is sufficient to consider the convergence of the corresponding series.

It is assumed that $a \in L_\infty(\mathbb{R})$ throughout the paper. Let $x = (x_1, x_2) \in D_2$ and let $\xi_x^a = (\xi_{x_1}^a, \xi_{x_2}^a)$ be the unique weak solution of the Cauchy problem [16, Corollary 3.11]

$$\begin{aligned}
d\xi_{x_k}^a(t) &= -a(\xi_{x_k}^a(t))dt + dw_k(t), \\
\xi_{x_k}^a(0) &= x_k, \quad k = 1,2,
\end{aligned} \tag{6}$$

where w_1, w_2 are independent standard Wiener processes started at 0. Define

$$\theta_x^a = \inf\{s \mid \xi_x^a \in \partial D_2\}. \tag{7}$$

Proposition 2. *Let $a \in L_\infty(\mathbb{R})$, $t > 0$. The density $p_t^{a,1}$ admits the representation*

$$p_t^{a,1}(u) = \lim_{\delta \to 0+} \delta^{-1} \mathrm{P}\left(\theta^a_{(u,u+\delta)} > t\right),$$

whenever the limit exits.

***Proof*:** As in the proof of Proposition 1, one gets, using the notion of the dual flow,

$$\begin{aligned}\mathrm{P}\left(X^a(\mathbb{R},t) \cap [u;u+\delta] \neq \emptyset\right) &= \mathrm{P}\left(X^{-a}(u+\delta,t) > X^{-a}(u,t)\right)\\ &= \mathrm{P}\left(\theta^a_{(u,u+\delta)} > t\right)\end{aligned}$$

for all $\delta > 0$. ■

The next example demonstrates the convergence of 1−point densities in the case of linear drift.

Example 1: Using the previous result one immediately gets $p_t^{0,1}(u) = \frac{1}{\sqrt{\pi t}}$. Let $a_c(x) = cx, c \neq 0$. The existence of the Arratia flow X^{a_c} follows from Remark 1. Moreover, using the arguments from [5, proof of Lemma 2.2], it is possible to establish the analogue of Proposition 2. Solving the SDE for $\xi^{a_c}_{u_2} - \xi^{a_c}_{u_1}$ explicitly and using the arguments from [5, proof of Lemma 2.2] one gets

$$p_t^{a_c,1}(u) = \sqrt{\frac{2}{\pi}} \frac{|c|^{1/2}}{\psi(t,c)}, \quad u \in \mathbb{R},$$

where

$$\psi(t,c) = \begin{cases} \left(\mathrm{e}^{2tc} - 1\right)^{1/2}, & c > 0,\ t > 0,\\ \left(1 - \mathrm{e}^{2tc}\right)^{1/2}, & c < 0,\ t > 0. \end{cases}$$

Thus $p_t^{0,1} = \lim_{c \to 0} p_t^{a_c,1}$ *a.e.* in $\mathbb{R}$.

Note that due to the domain D_2 being unbounded the problem (4)–(5) does not admit, in general, a unique solution. However, as usually, one expects the unique bounded solution W^a to have a probabilistic representation, that is, to satisfy $W^a(x,s) = \mathrm{P}(\theta^a_x > s)$. At the same time, in the case of a smooth enough the Duhamel principle [8, § 2.3.1.c] states

$$W^a(x,s) = \int_{D_2} \mathrm{d}y_0 g_s(x,y_0) + \int_0^s \mathrm{d}r \int_{D_2} \mathrm{d}y g_{s-r}(x,y) \left(-\nabla^a_y W^a(y,r)\right),$$

where

$$g_r(x,y) = \frac{1}{2\pi r}\left(\mathrm{e}^{-\frac{\|x-y\|^2}{2r}} - \mathrm{e}^{-\frac{\|x-y^*\|^2}{2r}}\right),$$

with $y^* = (y_2, y_1)$, is the transition density of the two-dimensional Wiener process killed when it reaches ∂D_2. Using this formal relation as a starting point we define, for any $a \in$

$L_\infty(\mathbb{R})$,

$$W^a(x,s) = \sum_{n\geq 0} W_n^a(x,s),$$

$$W_0^a(x,s) = \int_{D_2} \mathrm{d}y_0 g_s(x,y_0),$$

$$W_n^a(x,s) = (-1)^n \int_{\Delta_n(s)} \mathrm{d}r_1 \dots dr_n \int_{D_2^{n+1}} \mathrm{d}y_0 \dots \mathrm{d}y_n g_{s-r_n}(x,y_n) \prod_{j=1}^{n} \nabla_{y_j}^a g_{r_j - r_{j-1}}(y_j, y_{j-1}),$$

$$n \geq 1,$$

where hereinafter $\Delta_k(r) = \{u \in \mathbb{R}^k \mid 0 \leq u_1 \leq \dots \leq u_k \leq r\}, r > 0, k \geq 1$, so that

$$W_n^a(x,s) = -\int_0^s \mathrm{d}r_n \int_{D_2} \mathrm{d}y_n g_{s-r_n}(x,y_n) \nabla_{y_n}^a W_{n-1}^a(y_n, r_n), \quad n \geq 1.$$

Note that always

$$W_0^a((x_1,x_2),s) = \mathrm{P}\left(\theta^0_{(x_1,x_2)} > s\right) = \mathrm{P}\left(\sup_{r\leq s} w(r) < \frac{x_2 - x_1}{\sqrt{2}}\right)$$
$$= \sqrt{\frac{2}{\pi}} \int_0^{\frac{x_2-x_1}{\sqrt{2s}}} \mathrm{d}a\, \mathrm{e}^{-\frac{a^2}{2}},$$

where w is a standard Wiener process, and

$$\partial_{x_2} W_0^a((u,u),s) = -\partial_{x_1} W_0^a((u,u),s) = \frac{1}{\sqrt{\pi s}}, \quad s > 0.$$

The next lemma gathers technical results from calculus that are needed afterwards. In what follows $\overline{D}_2$ is the closure of D_2, the derivatives $\partial_{x_k}, k = 1,2$ on ∂D_2 are understood as one-sided derivatives and, for $y = (y_1, y_2)$,

$$G_a(y) = \min\left\{\sum_{j=1}^{2} |a(y_j)|,\ \|a\|_{L_\infty(\mathbb{R})}\right\}.$$

Lemma 1. (1) $\exists K, \gamma > 0\ \forall\, a \in L_\infty(\mathbb{R})\ \forall\, r > 0$

$$\left|\partial_{y_{1,k}} g_r(y_1,y_2)\right| \leq K r^{-3/2} \mathrm{e}^{-\gamma \frac{\|y_1-y_2\|^2}{r}} \text{ in } \overline{D}_2, \quad k = 1,2,$$
$$\left|\nabla_{y_1}^a g_r(y_1,y_2)\right| \leq K G_a(y_1) r^{-3/2} \mathrm{e}^{-\gamma \frac{\|y_1-y_2\|^2}{r}} \text{ a.e. in} \overline{D}_2,$$

where $y_1 = (y_{1,1}, y_{1,2})$;

(2) $\forall\, \alpha > 0\ \forall\, y \in \mathbb{R}^2$

$$\int_{D_2} \mathrm{d}y_0\, \mathrm{e}^{-\alpha\|y_0 - y\|^2} \leq \frac{\pi}{\alpha};$$

(3) *put* $r_0 = 0$; *then* $\forall\, s > 0\, \forall\, n \geq 1$

$$\int_{\Delta_n(s)} \mathrm{d}r_1 \ldots \mathrm{d}r_n (s - r_n)^{-1/2} \prod_{j=1}^{n} (r_j - r_{j-1})^{-1/2} = \frac{\pi^{\frac{n+1}{2}} s^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)},$$

$$\int_{\Delta_n(s)} \mathrm{d}r_1 \ldots \mathrm{d}r_n \prod_{j=1}^{n} (r_j - r_{j-1})^{-1/2} = \frac{2\pi^{\frac{n}{2}} s^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)};$$

where here and hereinafter Γ *is the gamma-function;*

(4) *for any* $n \geq 1$ *the derivatives* $\partial_{x_k} W_n^a, k = 1, 2$, *exist on* $\overline{D}_2 \times (0; +\infty)$, *and* $W_n^a, \partial_{x_k} W_n^a \in C(\overline{D}_2 \times (0; +\infty))$; *additionally,*

$$\sup_{x \in \overline{D}_2} \left| W_n^a(x, s) \right| \leq C_1 \frac{C_2^n \|a\|_{L_\infty(\mathbb{R})}^n s^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)},$$

$$\sup_{x \in \overline{D}_2} \left| \partial_{x_k} W_n^a(x, s) \right| \leq C_1 \frac{C_2^n \|a\|_{L_\infty(\mathbb{R})}^n s^{\frac{n-1}{2}}}{\Gamma\left(\frac{n}{2}\right)}, \quad s > 0, k = 1, 2, n \geq 1,$$

for some absolute positive constants C_1, C_2 *independent of s,x,n and a;*

(5) *for any* $t > 0$ *the series for* W^a *converges uniformly on* $\overline{D}_2 \times [0; t]$; *the derivatives* $\partial_{x_k} W^a, k = 1, 2$, *exist and are continuous on* $\overline{D}_2 \times (0; +\infty)$.

Proof: Denote the coordinates of the point y_2 by $y_{2,1}, y_{2,2}$, and put $y_2^* = (y_{2,2}, y_{2,1})$. Note that, for $k = 1, 2$,

$$\begin{aligned}
|\partial_{y_{1,k}} g_r(y_1, y_2)| &\leq \frac{1}{2\pi r^2} \left(\mathrm{e}^{-\frac{\|y_1 - y_2\|^2}{2r}} |y_{1,k} - y_{2,k}| + \mathrm{e}^{-\frac{\|y_1 - y_2^*\|^2}{2r}} |y_{1,k} - y_{2,k}^*| \right) \\
&\leq \frac{1}{2\pi r^2} \left(\mathrm{e}^{-\frac{\|y_1 - y_2\|^2}{2r}} \|y_1 - y_2\| + \mathrm{e}^{-\frac{\|y_1 - y_2^*\|^2}{2r}} \|y_1 - y_2^*\| \right) \\
&\leq \frac{C}{r^{3/2}} \left(\mathrm{e}^{-\frac{\|y_1 - y_2\|^2}{4r}} + \mathrm{e}^{-\frac{\|y_1 - y_2^*\|^2}{4r}} \right),
\end{aligned}$$

for some absolute constant C. Since $\mathrm{e}^{-\frac{\|y_1 - y_2^*\|^2}{4r}} \leq \mathrm{e}^{-\frac{\|y_1 - y_2\|^2}{4r}}$ for $y_1, y_2 \in \overline{D}_2$ this yields Item (1).

For Item (2), write

$$\int_{D_2} \mathrm{d}y_0\, \mathrm{e}^{-\alpha \|y_0 - y\|^2} \leq 2\pi \int_0^\infty \mathrm{d}\rho\, \rho \mathrm{e}^{-\alpha\rho^2} = \frac{\pi}{\alpha}.$$

Item (3) is a direct consequence of the integral representation given in [2, § 4.3] after the change of variables $r_1 = su_1, r_2 = s(u_1 + u_2), \ldots, r_n = s(u_1 + \ldots + u_n)$.[1]

Fix n and put $r_{n+1} = s, y_{n+1} = x$. Combining Items (1) and (2) we get

$$\begin{aligned}
\left|W_n^a(x,s)\right| &\le K^n \|a\|_{L_\infty(\mathbb{R})}^n \int_{\Delta_n(s)} \mathrm{d}r_1 \dots \mathrm{d}r_n \int_{D_2^{n+1}} \mathrm{d}y_0 \dots \mathrm{d}y_n g_{s-r_n}(x,y_n) \times \\
&\quad \times \prod_{j=1}^{n} \frac{\mathrm{e}^{-\gamma \frac{\|y_j - y_{j-1}\|^2}{r_j - r_{j-1}}}}{(r_j - r_{j-1})^{3/2}} \\
&\le K^n \|a\|_{L_\infty(\mathbb{R})}^n \left(\frac{\pi}{\gamma}\right)^n \int_{\Delta_n(s)} \mathrm{d}r_1 \dots \mathrm{d}r_n \prod_{j=1}^{n} (r_j - r_{j-1})^{-1/2}, \\
\left|\partial_{x_k} W_n^a(x,s)\right| &\le K^{n+1} \|a\|_{L_\infty(\mathbb{R})}^n \int_{\Delta_n(s)} \mathrm{d}r_1 \dots \mathrm{d}r_n \int_{D_2^{n+1}} \mathrm{d}y_0 \dots \mathrm{d}y_n \prod_{j=1}^{n+1} \frac{\mathrm{e}^{-\gamma \frac{\|y_j - y_{j-1}\|^2}{r_j - r_{j-1}}}}{(r_j - r_{j-1})^{3/2}} \\
&\le K^{n+1} \|a\|_{L_\infty(\mathbb{R})}^n \left(\frac{\pi}{\gamma}\right)^{n+1} \int_{\Delta_n(s)} \mathrm{d}r_1 \dots \mathrm{d}r_n \prod_{j=1}^{n+1} (r_j - r_{j-1})^{-1/2},
\end{aligned}$$

so the rest follows by calculus. ∎

In [19], a general version of the problem (4)–(5) in a bounded domain is studied in terms of weak (variational) solutions[2] so the smoothness of the solution follows by standard results. Since it is easy to see that neither $\partial_{x_k x_k} W_n^a$ nor $\partial_s W_n^a$ exists for $n \ge 1$ this is not a viable approach for us, and we use the hypoellipticity of an operator $\frac{1}{2}\Delta - \nabla_x^a$ for $a \in C^\infty(\mathbb{R}) \cap L_\infty(\mathbb{R})$ instead to check that the distribution solution W^a is sufficiently smooth.

The proof of the following lemma is trivial.

Lemma 2. *Let $M \in \mathcal{B}(\mathbb{R}^m)$ for some $m \in \mathbb{N}$, and $k \in L_1(M)$. Suppose functions $f_n \colon M \mapsto \mathbb{R}$, $n \ge 0$, are such that*

$$\sup_{n \ge 0} \|f_n\|_{L_\infty(M)} < \infty.$$

Suppose that one of the following conditions holds:

(1) $f_n \in L_1(M)$, $n \ge 0$, *and* $f_n \to f_0$, $n \to \infty$, *in* $L_1(M)$;
(2) $f_n \to f_0$, $n \to \infty$, *in* $L_\infty(M)$.

Then

$$\int_M \mathrm{d}y f_n(y) k(y) \to \int_M \mathrm{d}y f_0(y) k(y), \quad n \to \infty.$$

Proposition 3. *For all $n \ge 1$ in the sense of Schwartz distributions*

$$\partial_s W_n^a = \frac{1}{2} \Delta W_n^a - \nabla_x^a W_{n-1}^a$$

in $D_2 \times (0; \infty)$.

Proof: Suppose that $t > 0$ is fixed throughout the proof. We have, for $n \geq 1$,

$$W_n^a(x,s) = \int_0^s \int_{D_2} \mathrm{d}r\,\mathrm{d}y g_{s-r}(x,y) f_n(r,y), \quad x \in D_2,\ s \in (0;t),$$

where

$$f_n(r,y) = -\nabla_y^a W_{n-1}^a(y,r).$$

By Item 4 of Lemma 1

$$\sup_{n\geq 1} \operatorname*{ess\,sup}_{r\in(0;t),y\in D_2} |f_n(r,y)| \leq F r^{-\frac{1}{2}}, \tag{8}$$

for an absolute constant F. Consider for $h > 0$

$$\begin{aligned} W_n^a(x,s+h) - W_n^a(x,s) &= \int_0^s \int_{D_2} \mathrm{d}r\,\mathrm{d}y \left(g_{s+h-r}(x,y) - g_{s-r}(x,y)\right) f_n(r,y) \\ &\quad + \int_s^{s+h} \int_{D_2} \mathrm{d}r\,\mathrm{d}y g_{s+h-r}(x,y) f_n(r,y) \\ &= H_1(h,x,s) + H_2(h,x,s). \end{aligned} \tag{9}$$

Given a compactly supported in D_2 function $v \in C^\infty(D_2)$ we have, by the Fubini theorem, properties of the Gaussian density and the Green formula,

$$\begin{aligned} &h^{-1} \int_{D_2} \mathrm{d}x v(x) H_1(h,x,s) \\ &= \int_{D_2} \mathrm{d}x v(x) h^{-1} \int_0^s \int_{D_2} \mathrm{d}r\,\mathrm{d}y \int_{s-r}^{s+h-r} \mathrm{d}\tau\ \partial_\tau g_\tau(x,y) f_n(r,y) \\ &= \int_0^s \int_{D_2} \mathrm{d}r\,\mathrm{d}y \int_{s-r}^{s+h-r} \mathrm{d}\tau\ h^{-1} f_n(r,y) \int_{D_2} \mathrm{d}x \partial_\tau g_\tau(x,y) v(x) \\ &= \int_0^s \int_{D_2} \mathrm{d}r\,\mathrm{d}y \int_{s-r}^{s+h-r} \mathrm{d}\tau\ h^{-1} f_n(r,y) \int_{D_2} \mathrm{d}x \frac{1}{2} \Delta_x g_\tau(x,y) v(x) \\ &= -\frac{1}{2} \int_0^s \int_{D_2} \mathrm{d}r\,\mathrm{d}y \int_{s-r}^{s+h-r} \mathrm{d}\tau h^{-1} f_n(r,y) \int_{D_2} \mathrm{d}x\ \nabla_x g_\tau(x,y) \cdot \nabla_x v(x) \\ &= -\frac{1}{2} \int_{D_2} \int_0^s \int_{D_2} \mathrm{d}x\,\mathrm{d}r\,\mathrm{d}y \nabla_x v(x) \cdot \left(h^{-1} \int_r^{r+h} \mathrm{d}\tau\ f_n(s-r,y) \nabla_x g_\tau(x,y) \right), \end{aligned}$$

where $\cdot$ stands for the inner product in $\mathbb{R}^2$. For all r

$$h^{-1} \int_r^{r+h} \mathrm{d}\tau\, \partial_{x_k} g_\tau(x,y) \to \partial_{x_k} g_r(x,y), \quad k = 1,2.$$

Thus to prove

$$\begin{aligned} h^{-1} \int_{D_2} \mathrm{d}x v(x) H_1(h,x,s) &\to -\frac{1}{2} \int_{D_2} \int_0^s \int_{D_2} \mathrm{d}x\,\mathrm{d}r\,\mathrm{d}y \nabla_x v(x) \cdot f_n(r,y) \nabla_x g_{s-r}(x,y) \\ = -\frac{1}{2} \int_{D_2} \mathrm{d}x \nabla_x v(x) \cdot \nabla_x W_n^a(x,s), &\quad h \to 0+, \end{aligned} \tag{10}$$

by the means of dominated convergence theorem, it is sufficient to show that both families

$$\left\{(x,r,y)\mapsto h^{-1}\partial_{x_k}\nu(x)f_n(s-r,y)\int_r^{r+h}\mathrm{d}\tau\,\partial_{x_k}g_\tau(x,y)\mid h\in(0;1)\right\},$$
$$k=1,2,$$

are uniformly integrable on $\operatorname{supp}\nu\times(0;t)\times D_2$. For that, note that by Lemma 1 and (8) for almost all (r,y)

$$\left|h^{-1}f_n(s-r,y)\int_r^{r+h}\mathrm{d}\tau\;\partial_{x_k}g_\tau(x,y)\right|\le KFh^{-1}(s-r)^{-\frac12}\int_r^{r+h}\mathrm{d}\tau\,\tau^{-\frac32}\mathrm{e}^{-\gamma\frac{\|x-y\|^2}{\tau}},$$

where, for some positive C,

$$h^{-1}\int_r^{r+h}\mathrm{d}\tau\,\tau^{-\frac32}\mathrm{e}^{-\gamma\frac{\|x-y\|^2}{\tau}}\le\begin{cases}r^{-\frac32}\mathrm{e}^{-\gamma\frac{\|x-y\|^2}{r}}, & \frac23<\dfrac{r}{\gamma\|x-y\|^2},\\ (r+h)^{-\frac32}\mathrm{e}^{-\gamma\frac{\|x-y\|^2}{r+h}}, & \frac23>\dfrac{r+h}{\gamma\|x-y\|^2},\\ \dfrac{C}{\|x-y\|^3}, & \frac23\in\left[\dfrac{r}{\gamma\|x-y\|^2};\dfrac{r+h}{\gamma\|x-y\|^2}\right],\end{cases}$$

so that, for some $C_1,C_2>0$, all $\alpha\in(1;\frac43)$ and $\delta\in(0;1)$, we have, due to Item 2 of Lemma 1, for almost all r

$$\begin{aligned}&\int_{D_2}\mathrm{d}y\left|f_n(s-r,y)h^{-1}\int_r^{r+h}\mathrm{d}\tau\,\partial_{x_k}g_\tau(x,y)\right|^\alpha\\
&\le\frac{C_1}{(s-r)^{\frac\alpha2}}\int_{D_2}\mathrm{d}y\left\{r^{-\frac{3\alpha}{2}}\mathrm{e}^{-\frac{\alpha\gamma\|x-y\|^2}{r}}+(r+h)^{-\frac{3\alpha}{2}}\mathrm{e}^{-\frac{\alpha\gamma\|x-y\|^2}{r+h}}\right.\\
&\quad\left.+\frac{1}{\|x-y\|^{3\alpha}}\amalg\!\!\amalg\left[\frac23\in\left[\frac{r}{\gamma\|x-y\|^2};\frac{r+h}{\gamma\|x-y\|^2}\right]\right]\right\}\\
&\le\frac{C_1}{(s-r)^{\frac\alpha2}}\left\{\frac{\pi}{\alpha\gamma}\left(r^{1-\frac{3\alpha}{2}}+(r+h)^{1-\frac{3\alpha}{2}}\right)\right.\\
&\quad\left.+2\pi\int_0^\infty\mathrm{d}\rho\;\rho^{1-3\alpha}\amalg\!\!\amalg\left[\frac{r}{\rho^2}\le\frac{2\gamma}{3}\le\frac{r+h}{\rho^2}\right]\right\}\\
&\le\frac{C_1\left(\frac{\pi}{\alpha\gamma}+2\pi\right)}{(s-r)^{\frac\alpha2}}\left\{\left(r^{1-\frac{3\alpha}{2}}+(r+h)^{1-\frac{3\alpha}{2}}\right)+\int_{\sqrt{\frac{3r}{2\gamma}}}^{\sqrt{\frac{3(s+1)}{2\gamma}}}\mathrm{d}\rho\;\rho^{1-3\alpha}\right\}\\
&\le\frac{C_1\left(\frac{\pi}{\alpha\gamma}+2\pi\right)}{(s-r)^{\frac\alpha2}}\left\{\left(r^{1-\frac{3\alpha}{2}}+(r+h)^{1-\frac{3\alpha}{2}}\right)+\left(\frac{3r}{2\gamma}\right)^{1-\frac{3\alpha+\delta}{2}}\int_0^{\sqrt{\frac{3(s+1)}{2\gamma}}}\mathrm{d}\rho\;\rho^{-1+\delta}\right\}\\
&\le\frac{C_2}{(s-r)^{\frac\alpha2}}\left\{\left(r^{1-\frac{3\alpha}{2}}+(r+h)^{1-\frac{3\alpha}{2}}\right)+r^{1-\frac{3\alpha+\delta}{2}}(s+1)^{\frac\delta2}\right\}.\end{aligned}$$

Choosing δ in such a way that $1-\frac{3\alpha+\delta}{2} > -1$ gives, for some $C_3 > 0$,

$$\begin{aligned}
&\sup_{x\in D_2}\int_0^s\int_{D_2} \mathrm{d}r\,\mathrm{d}y \left| h^{-1} f_n(s-r,y)\int_r^{r+h} \mathrm{d}\tau\ \partial_{x_k} g_\tau(x,y)\right|^\alpha \\
&\quad \le C_2\left\{2\int_0^s \mathrm{d}r \frac{r^{1-\frac{3\alpha}{2}}}{(s-r)^{\frac{\alpha}{2}}} + (s+1)^{\frac{\delta}{2}}\int_0^s \mathrm{d}r\ \frac{r^{1-\frac{3\alpha+\delta}{2}}}{(s-r)^{\frac{\alpha}{2}}}\right\} \\
&\quad \le C_3\int_0^s \mathrm{d}r \frac{r^{1-\frac{3\alpha+\delta}{2}}}{(s-r)^{\frac{\alpha}{2}}} \\
&\quad < +\infty,
\end{aligned}$$

which yields (10).

For H_2, we have, in terms of the process ξ_x^a and the stopping time θ_x^a defined in (6)–(7)

$$h^{-1}H_2(h,x,s) = h^{-1}\int_s^{s+h} \mathrm{d}r k_{x,s}(h,r),$$

where

$$k_{x,s}(h,r) = \mathrm{E} f_n\left(r,\xi_x^a(s+h-r)\right)\mathrm{I\!I}\left[\theta_x^a > s+h-r\right].$$

Since a is bounded, the Novikov condition holds on $(0;t)$ so by the Girsanov theorem

$$k_{x,s}(h,r) = \mathrm{E} f_n\left(r,\xi_x^0(s+h-r)\right)\mathrm{I\!I}\left[\theta_x^0 > s+h-r\right]\mathcal{E}_t^a,$$

where $\mathcal{E}_t^a$ is the corresponding stochastic exponential. The family of random variables

$$\left\{f_n\left(r,\xi_x^0(s+h-r)\right)\mathrm{I\!I}\left[\theta_x^0 > s+h-r\right]\mathcal{E}_t^a \mid (r,h)\in A\right\},$$

where $A = \{(u_1,u_2) \mid u_1 \in [s;s+u_2], u_2 \in [0;1]\}$, is uniformly integrable, so since the distribution of θ_x^0 has no atoms the function $k_{x,s}$ is uniformly continuous on A for any (x,s). Fix $\epsilon > 0$. Then there exists h_0 such that

$$h^{-1}\int_s^{s+h} \mathrm{d}r|k_{x,s}(h_0,r) - k_{x,s}(h,r)| \le \epsilon$$

whenever $h \le h_0$. By the Lebesgue differentiation theorem for fixed h_0

$$h^{-1}\int_s^{s+h} \mathrm{d}r k_{x,s}(h_0,r) \to k_{x,s}(h_0,s), \quad h\to 0+,$$

where

$$k_{x,s}(h_0,s) \to k_{x,s}(0,s) = f_n(s,x), \quad h_0 \to 0+.$$

Therefore

$$\lim_{h\to 0+} h^{-1}H_2(h,x,s) = \lim_{h_0\to 0+} k_{x,s}(h_0,s) = f_n(s,x) = -\nabla_x^a W_{n-1}^a(x,s).$$

Consider the analogue of (9) for a negative increment: for $h > 0$

$$\begin{aligned} W_n^a(x, s-h) - W_n^a(x,s) &= \int_0^{s-h} \int_{D_2} \mathrm{d}r\,\mathrm{d}y \left(g_{s-h-r}(x,y) - g_{s-r}(x,y)\right) f_n(r,y) \\ &\quad + \int_{s-h}^{s} \int_{D_2} \mathrm{d}r\,\mathrm{d}y g_{s-h-r}(x,y) f_n(r,y) \\ &= \widetilde{H}_1(h,x,s) + \widetilde{H}_2(h,x,s), \end{aligned}$$

and the reasoning for $\widetilde{H}_1$ and $\widetilde{H}_2$ is the same as for their counterparts H_1 and H_2, which concludes the proof. ■

Theorem 2. *Let $a \in L_\infty(\mathbb{R})$. For all $s > 0$ and $x \in D_2$*

$$W^a(x,s) = \mathrm{P}\left(\theta_x^a > s\right)$$

where θ_x^a is defined in (7).

***Proof*:** Assume $a \in C^\infty(\mathbb{R})$ additionally. It follows from Proposition 3 and the completeness of the space of Schwartz distributions that in the sense of Schwartz distributions

$$\partial_s W^a = \frac{1}{2}\Delta W^a - \nabla_x^a W^a$$

in $(0;\infty) \times D_2$. Since the operator $\frac{1}{2}\Delta - \nabla_x^a$ is hypoelliptic, $W^a \in C^\infty(D_2 \times (0;\infty))$ [22, Theorem 3.4.1]. By Item 4 of Lemma 1, $W^a \in C(\overline{D}_2 \times (0;\infty)) \cap C(D_2 \times [0;\infty))$. Since

$$g_r((u,u),y) = 0, \quad r \geq 0,\ u \in \mathbb{R},\ y \in \mathbb{R}^2,$$

W^a satisfies the Dirichlet boundary condition as a function. To show

$$W^a(x,s) = \mathrm{P}\left(\theta_x^a > s\right), \tag{11}$$

one proceeds as follows.

Let

$$K_n = \left\{ x \in D_2 \mid \|x\| \leq n,\ \inf_{y \in \partial D_2} \|x - y\| \geq \frac{1}{n} \right\}, \quad n \geq 1,$$

and define, for fixed $T > 0$,

$$\begin{aligned} \theta_{t,x;T}^n &= T \wedge \inf\left\{ s \geq t \mid \xi_{t,x}^a(s) \notin K_n \right\}, \\ \theta_{t,x} &= \inf\left\{ s \geq t \mid \xi_{t,x}^a(s) \notin D_2 \right\}, \quad t > 0, x \in D_2, \end{aligned}$$

where, analogously to (6), for a standard Wiener process w

$$\begin{aligned} d\xi_{t,x}^a(s) &= -a(\xi_{t,x}^a(s))ds + dw(s), \quad s \geq t, \\ \xi_{t,x}^a(t) &= x. \end{aligned}$$

The standard reasoning via the Itó formula and the exhaustion method (e.g. [12, § 2.2] or [13, Ch.8, § 5]) gives, for any fixed $h > 0, t \in (0; T)$,

$$\mathrm{E}\, W^a\left(h + T - \theta^n_{t,x;T}, \xi^a_{t,x}(\theta^n_{t,x;T})\right) = W^a(h + T - t, x), \quad n \geq 1, x \in D_2. \tag{12}$$

Since $W^a \in C(D_2 \times [0; \infty))$

$$W^a(h + T - t, x) \to W^a(T - t, x), \quad h \to 0+.$$

We have

$$\mathrm{E}\, W^a\left(h + T - \theta^n_{t,x;T}, \xi^a_{t,x}(\theta^n_{t,x;T})\right) = \mathrm{E}\, W^a\left(h + T - t - \theta^n_{0,x;T-t}, \xi^a_{0,x}(\theta^n_{0,x;T-t})\right).$$

The process $\xi^a_{0,x}$ is bounded on $[0; T - t]$ with probability 1. On the set $\{\theta_{0,x} \leq T - t\}$

$$W^a\left(h + T - t - \theta^n_{0,x;T-t}, \xi^a_{0,x}(\theta^n_{0,x;T-t})\right) \to W^a\left(h + T - t - \theta_{0,x}, \xi^a_{0,x}(\theta_{0,x})\right) = 0,$$
$$n \to \infty,$$

since $W^a \in C(\overline{D}_2 \times (0; \infty))$. On the set $\{\theta_{0,x} > T - t\}$

$$W^a\left(h + T - t - \theta^n_{0,x;T-t}, \xi^a_{0,x}(\theta^n_{0,x;T-t})\right) \to W^a\left(h, \xi^a_{0,x}(T - t)\right), \quad n \to \infty,$$

and, since $\xi^a_{0,x}(T - t) \in D_2$ now and $W^a \in C(D_2 \times [0; \infty))$,

$$W^a\left(h, \xi^a_{0,x}(T - t)\right) \to W^a(0, x) = 1, \quad h \to 0+.$$

Returning to (12), passing to limit w.r.t n and h and introducing $s = T - t > 0$, we get

$$W^a(x, s) = \mathrm{E}\, \mathbb{1}\left[\theta_{0,x} > s\right],$$

which coincides with (11).

Consider the case $a \in L_\infty$. Let η be a mollifier and put $a_n = a * n\eta(\frac{\cdot}{n}), n \geq 1$. Additionally put $a_0 = a$. Then $a_n \in C^\infty(\mathbb{R})$, $|a_n| \leq |a|, n \geq 1$, and $a_n \to a, n \to \infty$, in $L_1(\mathbb{R})$. Consider $s \in (0; t)$ for a fixed t. Then the Novikov condition

$$\sup_{n \geq 0} \mathrm{E} \exp\left\{\frac{1}{2}\int_0^t \mathrm{d}s \sum_{j=1}^2 a_n^2\left(\xi^a_{x_j}(s)\right)\right\} < \infty,$$

holds, where $\xi^a_x = \xi^a_{0,x}$. Thus, by the Girsanov theorem,

$$\mathrm{P}\left(\theta^{a_n}_x > s\right) = \mathrm{E}\, \varkappa \mathcal{E}_n,$$

where

$$\varkappa = \mathbb{1}\left[\theta^0_x > s\right],$$
$$\mathcal{E}_n = \exp\left\{\sum_{j=1}^2 \int_0^t a_n\left(\xi^0_{x_j}(r)\right) \mathrm{d}\xi^0_{x_j}(r) - \frac{1}{2}\sum_{j=1}^2 \int_0^t \mathrm{d}r a_n^2\left(\xi^0_{x_j}(r)\right)\right\},$$
$$n \geq 0.$$

Since $a \in L_\infty(\mathbb{R})$ the Cauchy inequality implies that the sequence $\{\varkappa\mathcal{E}_n\}_{n\geq 1}$ is uniformly integrable. Consequently, in order to prove

$$\mathrm{P}\left(\theta_x^{a_n} > s\right) = \mathrm{E}\,\varkappa\mathcal{E}_n \to \mathrm{E}\,\varkappa\mathcal{E}_0 = \mathrm{P}\left(\theta_x^{a} > s\right), \quad n \to \infty, \tag{13}$$

it is sufficient to check that $\mathcal{E}_n \to \mathcal{E}_0$ in probability. For that, consider, for fixed j,

$$\mathrm{E}\left|\int_0^t \mathrm{d}r a_n^2\left(\xi_{x_j}^0(r)\right) - \int_0^t \mathrm{d}r a_0^2\left(\xi_{x_j}^0(r)\right)\right| \leq \int_0^t \mathrm{d}r\,\mathrm{E}\left|a_n^2\left(\xi_{x_j}^0(r)\right) - a_0^2\left(\xi_{x_j}^0(r)\right)\right|. \tag{14}$$

Since for every s the sequence

$$\left\{ u \mapsto \frac{\left|a_n^2(u) - a_0^2(u)\right| \mathrm{e}^{-\frac{\|u-x_j\|^2}{2s}}}{s} \mid n \geq 1 \right\}$$

is uniformly integrable w.r.t. the Lebesgue measure in $\mathbb{R}$; the dominated convergence theorem implies

$$\mathrm{E}\left|a_n^2\left(\xi_{x_j}^0(r)\right) - a_0^2\left(\xi_{x_j}^0(r)\right)\right| \to 0, \quad n \to \infty,$$

for all $s \in (0;t)$. Using the dominated convergence theorem one more time in (14) we obtain

$$\int_0^t \mathrm{d}r a_n^2\left(\xi_{x_j}^0(r)\right) \to \int_0^t \mathrm{d}r a_0^2\left(\xi_{x_j}^0(r)\right), \quad n \to \infty,$$

in probability. Analogously, one proves

$$\int_0^t a_n\left(\xi_{x_j}^0(r)\right)\,\mathrm{d}\xi_{x_j}^0(r) \to \int_0^t a_0\left(\xi_{x_j}^0(r)\right)\,\mathrm{d}\xi_{x_j}^0(r), \quad n \to \infty,$$

in probability, which finishes the proof of (13).

Thus

$$W^{a_n}(x,s) = \mathrm{P}\left(\theta_x^{a_n} > s\right) \to \mathrm{P}\left(\theta_x^{a} > s\right), \quad n \to \infty.$$

Therefore, it is left to show

$$W^{a_n}(x,s) \to W^{a}(x,s), \quad n \to \infty. \tag{15}$$

Note that by Item 4 of Lemma 1 for all $s \in (0;t)$ and $x \in D_2$

$$\sup_{n\geq 1}\left|\sum_{m\geq M} W_m^{a_n}(x,s) - \sum_{m\geq M} W_m^{a}(x,s)\right| \to 0, \quad M \to \infty. \tag{16}$$

For all $m \geq 1$ by Lemma 1

$$\begin{aligned}
&\left| \prod_{j=1}^{m} \nabla_{y_j}^{a} g_{r_j - r_{j-1}}(y_j, y_{j-1}) - \prod_{j=1}^{m} \nabla_{y_j}^{a_n} g_{r_j - r_{j-1}}(y_j, y_{j-1}) \right| \\
&\quad \leq \sum_{k=1}^{m} \left| \prod_{j=1}^{k-1} \nabla_{y_j}^{a_n} g_{r_j - r_{j-1}}(y_j, y_{j-1}) \, \nabla_{y_k}^{a_n - a} g_{r_k - r_{k-1}}(y_k, y_{k-1}) \times \right. \\
&\qquad \left. \times \prod_{j=k+1}^{m} \nabla_{y_j}^{a} g_{r_j - r_{j-1}}(y_j, y_{j-1}) \right| \\
&\quad \leq K^m \|a\|_{L_\infty(\mathbb{R})}^{m-1} \prod_{j=1}^{m} \frac{\mathrm{e}^{-\gamma \frac{\|y_1 - y_2\|^2}{r_j - r_{j-1}}}}{(r_j - r_{j-1})^{3/2}} \sum_{k=1}^{m} \sum_{j=1}^{2} |a_n(y_{k,j}) - a(y_{k,j})|,
\end{aligned}$$

where the product over the empty set is equal to 1 by definition. Therefore, due to (16) and the definition of W^a it is sufficient to show that

$$\int_{\Delta_m(s)} \mathrm{d}r_1 \ldots \mathrm{d}r_m \int_{D_2^{m+1}} \mathrm{d}y_0 \ldots \mathrm{d}y_m \prod_{j=1}^{m} \frac{\mathrm{e}^{-\gamma \frac{\|y_1 - y_2\|^2}{r_j - r_{j-1}}}}{(r_j - r_{j-1})^{3/2}} |a_n(y_{k,j}) - a(y_{k,j})| \to 0,$$

$n \to \infty$,

for fixed m and j. Setting $f_n(y_{k,j}) = |a_n(y_{k,j}) - a(y_{k,j})|$ in Lemma 2 and applying Items 2 and 3 of Lemma 1 finishes the proof. We omit the details. ■

Theorem 3. *Let $a \in L_\infty(\mathbb{R})$. Then for any $t > 0$ the density $p_t^{a,1}$ has a continuous version*

$$p_t^{a,1}(x) = \partial_{x_2} W^a((x,x),t) = \sum_{n \geq 0} \partial_{x_2} W_n^a((x,x),t).$$

Proof: Combining Proposition 2, Theorem 2 and the fact that $W^a((x,x),t) = 0$ we get

$$\begin{aligned}
p_t^{a,1}(x) &= \lim_{\delta \to 0+} \delta^{-1} \, \mathrm{P}\left(\theta^a_{(x,x+\delta)} > t\right) \\
&= \lim_{\delta \to 0+} \delta^{-1} \left(W^a((x, x+\delta), t) - W^a((x,x),t) \right),
\end{aligned}$$

so the conclusion follows by Lemma 1 and calculus. ■

The next result is stated for a continuous version of the 1−dimensional density.

Theorem 4. *Assume $a_n \in L_\infty(\mathbb{R})$, $n \geq 0$; $\sup_{n \geq 0} \|a_n\|_{L_\infty(\mathbb{R})} < \infty$. Let one of the following conditions hold:*

(1) $a_n \to a_0, n \to \infty$, *in* $L_\infty(\mathbb{R})$;
(2) $a_n \in L_1(\mathbb{R}), n \geq 0$, *and* $a_n \to a_0, n \to \infty$, *in* $L_1(\mathbb{R})$.

Then for any $t > 0$ and all $x \in \mathbb{R}$

$$p_t^{a_n,1}(x) \to p_t^{a_0,1}(x), \quad n \to \infty.$$

Proof: By Theorem 3, the conclusion of the theorem is equivalent to

$$\sum_{k\geq 0} \partial_{x_2} W_k^{a_n}((x,x),t) \to \sum_{k\geq 0} \partial_{x_2} W_k^{a_0}((x,x),t), \quad n \to \infty.$$

Since $\sup_{n\geq 0} \|a_n\|_{L_\infty(\mathbb{R})} < \infty$, one applies the same reasoning as in the proof of (15), utilizing the uniform estimates for $\partial_{x_k} W^{a_n}(x,s)$ of Item 4 in Lemma 1, and uses both statements of Lemma 2 subsequently. We omit the details. ■

Notes

1. The authors are grateful to a colleague for providing the reference [2].
2. The statement of [19, Theorem 2.7] contains a misprint: $Q(t,x)$ should be equal $\mathrm{P}(\tau_A > t)$.

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